Sankhyā: The Indian Journal of Statistics
1998, Volume 60, Series B, Pt. 2, pp. 331-348

ORDERING AND ASYMPTOTIC PROPERTIES OF RESIDUAL INCOME DISTRIBUTIONS

By F. BELZUNCE
J. CANDEL
and
J.M. RUIZ
University of Murcia, Spain

SUMMARY. In this paper we study the connection between new classes of distributions and the study of poverty and affluence of income distributions. Further it is shown that for these classes the upper and lower tails can be fitted by the Pareto and power models of distributions. We conclude with the estimation of inequality in the power model.

1. Introduction

The study of poverty and affluence may be based on the concept of residual incomes. Given a continuous, non negative random variable X, which represents the income of a society or community, with distribution function $F_X(x)$, a poverty line ω and an affluence line θ, where $0< \omega < \theta < +\infty$ are considered; such that $F_X(\omega)$ represents the proportion of the poor and $1 - F_X(\theta)$ represents the proportion of the rich. The affluence and poverty are then quantified in terms of the proportion of the rich and poor people and their income inequality (see Sen (1976), Takayama (1979) and Sen (1986), (1988)). The Gini coefficient is a measure that has been most widely used in the study of income inequality and is defined as twice the area between the Lorenz curve and the 'egalitarian line' $x = y$, where the Lorenz curve is defined by (Kakwani (1980))

$$L_X(p) = \frac{\int_0^p F_X^{-1}(t)dt}{E[X]}, \ p \in [0, 1].$$

Paper received. August 1994; revised January 1998.
AMS (1991) *subject classification.* Primary 62E10, 62F10, secondary 62F99, 62P20.
Key words and phrases. Classes of distributions, orderings of distributions, left and right truncated distributions, inequality measures, power distribution, maximum likelihood estimators, convergence in distribution.

The income distribution of the poor for a poverty line ω, is the right truncated income distribution at ω,

$$X_r(\omega) = \{X | X \leq \omega\};$$

and its distribution function is given by

$$F_{X_r(\omega)}(x) = \begin{cases} \frac{F_X(x)}{F_X(\omega)} & x \leq \omega \\ 1 & x \geq \omega \end{cases}.$$

The *income gap ratio* of the poor people is defined as

$$\beta(\omega) = 1 - \frac{E[X_r(\omega)]}{\omega}; \quad \ldots(1)$$

and the *Gini coefficient* of the income distribution among the poor is defined as

$$G(\omega) = 1 - 2\int_0^1 L_{X_r(\omega)}(p)dp; \quad \ldots(2)$$

where $L_{X_r(\omega)}(p)$, is the Lorenz curve for the poor.

In the literature several poverty indices have been considered, based on $F_X(\omega)$, $\beta(\omega)$ and $G(\omega)$ (see Sen (1976), Takayama (1979) and Sen (1986)).

For the income distribution of the affluent people we get the left truncated distribution at θ, $X_l(\theta) = \{X | X > \theta\}$, with distribution function given by

$$F_{X_l(\theta)}(x) = \begin{cases} 0 & x \leq \theta \\ \frac{F_X(x) - F_X(\theta)}{1 - F_X(\theta)} & x \geq \theta \end{cases}.$$

As a measure of the income gap ratio among the rich we consider (Sen (1988))

$$\beta^*(\theta) = 1 - \theta E[X_l(\theta)]^{-1}; \quad \ldots(3)$$

while as a measure of inequality among the rich people we consider the *harmonic Gini coefficient*, defined by (Sen (1988))

$$G^*(\theta) = 1 - 2\int_0^1 L_{X_l(\theta)^{-1}}(p)dp. \quad \ldots(4)$$

In a similar way, several affluence indices are defined in terms of 1-$F_X(\theta)$, $\beta^*(\theta)$ and $G^*(\theta)$ (see Sen (1988)).

Bhattacharjee and Krishnaji (1981) and Bhattacharjee (1988) introduced the indices

$$I_1 = \frac{\lim_{\theta \to \infty} E[X_l(\theta)] - \theta}{E[X]}$$

and

$$I_2 = \lim_{\theta \to \infty} \frac{E\left[X_l(\theta)\right]}{\theta}; \qquad \ldots (5)$$

to measure inequality in the context of distribution of landholding. Their point of departure is the consideration of gamma and log-gamma models for the distribution of landholding, which belong to some classes of distributions that are well known in the context of lifetime distributions and reliability studies. Further indices are considered by Bhattacharjee (1993). The relationship between some notions which are common to reliability theory and economics have received attention in the literature. Singh and Maddala (1976) proposed a new model of income distribution based on the concept of proportionate failure rate. Chandra and Singpurwalla (1981) show the relationships between the Lorenz curve and the Gini index and some concepts of reliability theory such as the total time on test transform and the mean residual life. Further results are given by Klesfjö (1984), Kochar (1989) and Pham and Turkkan (1994) for the reliability interpretations of some concepts from economics. In the context of reliability the concept of ageing plays a central role. One of the most important approaches to the study of ageing is based on the concept of *additional residual life at time t*. Some classes of distributions which reflect the ageing of a lifetime unit, are given in terms of this random variable (Bryson and Siddiqui (1969) and Barlow and Proschan (1975)). In the next section we consider the left and right proportional residual incomes, in terms of which we define some classes of distributions and show the connection between these classes and the measures previously mentioned. In Section 3 we study the asymptotic behavior of the proportional residual incomes for high and low levels. In particular we see that for upper and lower tails of the residual income we can fit the Pareto and the power distribution respectively. This last fact has lead us in Section 4, to study the estimation of the Lorenz curve and the Gini index in the power model. The study of estimation of inequality in the Pareto model has been considered by Moothathu ((1985a) and (1990)).

2. Left and Right Residual Incomes

Given a non negative random variable which represents the lifetime of a unit or a system; the additional residual life at time t is defined as $X_l(t) - t$. The comparison at different times, in some statistical sense, of the additional residual life leads to the definition of some well known classes of distributions (IFR, DFR, DMRL, IMRL, etc.). In a similar way we can consider the random variable

$$X_l''(\theta) = \frac{X_l(\theta)}{\theta};$$

and in the context of income distribution it represents the proportional income to θ, for incomes greater than θ. We will call this the *left proportional residual* *income at level* θ. As in the case of the additional residual life we can consider the following class.

DEFINITION 1. A non negative random variable X, is said to be IPFR (increasing proportional failure rate) if $X_l''(\theta) >_{ST} X_l''(\theta')$, for every $\theta < \theta'$.

The order relation $>_{ST}$ between two income distributions is defined as in Marshall and Olkin (1979), viz.,

$$X >_{ST} Y \Longleftrightarrow 1 - F_X(x) \geq 1 - F_Y(x), \text{ for every } x,$$

and is characterized as

$$X >_{ST} Y \Longleftrightarrow E[\phi(X)] \geq E[\phi(Y)], \qquad \ldots (6)$$

for every increasing function ϕ. The previous ordering provides the following interpretation of the IPFR class. The variable X is IPFR if the probability to obtain an income greater than x, proportional to the affluence line θ, is decreasing in θ. If the random variable X has a probability density function f_X, then the IPFR class can be characterized easily in terms of the proportional failure rate, $xr_X(x)$, where $r_X(x) = f_X(x)/(1 - F_X(x))$ is known as the *failure rate* (Barlow and Proschan (1975));

$$X \text{ is IPFR } \Longleftrightarrow xr_X(x), \text{ is increasing in } x.$$

The proportional failure rate has been used to describe the income distribution; in particular Singh and Maddala (1976) derived a model of income distribution from an increasing and bounded proportional failure rate, given by

$$xr_X(x) = \alpha x^{\beta}(1 - F_X(x))^{\gamma}, \ \alpha, \ \beta, \gamma > 0.$$

Solving this equation the distribution function is given by

$$F_X(x) = 1 - (1 + ax^b)^{-c}, \ x \geq 0;$$

where $a = \alpha(\gamma - 1)/(\beta + 1) > 0$, $b = \beta + 1 > 0$ and $c = 1/(\gamma - 1) > 0$.

We can also define a new class in terms of the *mean left proportional residual income* (MLPRI), viz.,

$$e_{X_l''}(\theta) = E[X_l''(\theta)] = E\left[\frac{X}{\theta}\middle| X > \theta\right].$$

DEFINITION 2. Let X be a non negative random variable. X is said to be DMLPRI (decreasing mean proportional residual income) if $E[X_l''(\theta)]$ is decreasing in θ.

By definitions 1 and 2, and (6) it is clear that

$$\text{IPFR} \Longrightarrow \text{DMLPRI}.$$

As we can see, the index I_2 given in (5) can be written in terms of the MLPRI as

$$I_2 = \lim_{\theta \to +\infty} e_{X_l''}(\theta).$$

Bhattacharjee and Krishnaji (1981) and Bhattacharjee (1993) study conditions for the existence and values of I_2. In terms of the previous classes we next give the following results for the measure I_2.

PROPOSITION 3. *Let X be a non negative random variable, with density function and finite mean, then*

a) If X is IPFR or DMLPRI then $1 \leq I_2 < +\infty$.

b) If X is IPFR or DMLPRI then $1 < I_2 < \infty$ if and only if $1 < \lim_{x \to +\infty} x r_X(x) < +\infty$.

PROOF a) If X is IPFR or X is DMLPRI then we get that the MLPRI is a decreasing function and bounded below, therefore the limit $\lim_{\theta \to +\infty} e_{X_l''}(\theta)$ exists and is finite.

b) We get that (lemma 3.1, Bhattacharjee and Krishnaji (1981))

$$I_2 = \lim_{x \to +\infty} \frac{x r_X(x)}{x r_X(x) - 1};$$

and the result follows easily. □

The previous classes are related to a decreasing ordering in inequality of the left residual incomes. The comparison of inequality at different levels of affluence is based on the following partial orderings of distributions.

In the context of income distributions some orderings have been defined to compare the concentration and the inequality between income distributions. It is said that the random variable Y is greater than X in the star shaped ordering ($X <_* Y$) if (Arnold (1987))

$$\frac{F_Y^{-1}(p)}{F_X^{-1}(p)}, \text{ is increasing in } p \in (0,1). \qquad \ldots(7)$$

where $F_X^{-1}(p) = \inf \{x : F_X(x) \geq p\}$.

Also the Lorenz curve has been used to compare the concentration between income distributions. It is said that the random variable Y is greater than the random variable X in the Lorenz sense ($X <_L Y$) if (Arnold (1987))

$$L_X(p) \geq L_Y(p), \forall p \in [0,1].$$

Taillie (1981) gives some characterizations of the Lorenz ordering. In particular we will use the following results:

$$\begin{aligned} X <_L Y &\iff \tfrac{X}{\mu_X} <_{ST2} \tfrac{Y}{\mu_Y} \\ &\iff E\left[\varphi(X/\mu_X)\right] \leq E\left[\varphi(Y/\mu_Y)\right], \text{ for every convex function } \varphi, \end{aligned} \qquad \ldots(8)$$

where the ST2 ordering is defined as (Rolski (1976))

$$X <_{ST2} Y \Longleftrightarrow \int_x^{+\infty} \overline{F}_X(t)dt \leq \int_x^{+\infty} \overline{F}_Y(t)dt, \text{ for all } x;$$

$\overline{F}_X(x) = 1 - F_X(x)$ being the survival function of X.

In terms of these orderings the previous classes are characterized, for an absolutely continuous random variable, as follows

$$a)\ X \text{ is IPFR} \Longleftrightarrow X_l(\theta) >_* X_l(\theta'),\ \forall\theta < \theta';$$

and

$$b)\ X \text{ is DMLPRI} \Longleftrightarrow X_l(\theta) >_L X_l(\theta'),\ \forall\theta < \theta'.$$

The results *a*) and *b*) have been given by Belzunce, Candel and Ruiz (1995) and Ord, Patil and Taillie (1983) respectively. Therefore we get that an increasing specification of the affluent line θ, gives a decreasing inequality between the rich people.

DEFINITION 4. Let X be a non negative random variable, then the right residual income at level ω, is defined as

$$X_r''(\omega) = \left\{ \frac{X}{\omega} \middle| X \leq \omega \right\}.$$

In terms of this random variable we define the following class of distributions.

DEFINITION 5. A non negative random variable X is said to be decreasing reversed proportional failure rate (DRPFR) if $X_r''(\omega)$ is stochastically decreasing in ω.

For the absolute continuous case we can characterize the previous classes in terms of the following function.

DEFINITION 6. Let X be an absolutely continuous, non negative random variable, with distribution function F_X and density function f_X. The reversed proportional failure rate (RPFR) of X is defined to be the function

$$x\frac{f_X(x)}{F_X(x)}.$$

The previous function is the income elasticity of the cdf $F_X(x)$. We obtain now the following result.

PROPOSITION 7. *Let X be an absolute continuous, non negative random variable. Then X is DRPFR if and only if the reversed proportional failure rate is decreasing .*

PROOF. We can get the following chain of equivalences:

$$X \text{ is DRPFR} \Longleftrightarrow \frac{\partial}{\partial\omega} F_{X_r^{\mu}(\omega)}(x) \geq 0,\ \forall 0 \leq x \leq 1$$

$$\Longleftrightarrow \omega x \frac{f_X(\omega x)}{F_X(\omega x)} \geq \omega \frac{f_X(\omega)}{F_X(\omega)};\ \forall \omega > 0, \forall 0 \leq x \leq 1;$$

so $x f_X(x)/F_X(x)$ is decreasing. □

The reversed proportional failure rates have also been used, to derive a model of income distribution. Dagum (1977) specifies a decreasing reversed proportional failure rate, given by

$$x\frac{f_X(x)}{F_X(x)} = \beta\delta\left[1 - F_X(x)^{1/\beta}\right],\ \beta > 0,\ \beta\delta > 1.$$

The solution of this equation is given by

$$F_X(x) = \left[1 + \lambda x^{-\delta}\right]^{-\beta},\ \lambda > 0;$$

and is known as the Dagum model type I. Dagum (1977) observes that the empirical income distributions show systematically in both developed and developing countries, a decreasing income elasticity of the cdf $F_X(x)$ as a function of $F_X(x)$; and hence of x. Therefore the empirical income distributions belong to the class DRPFR.

Bhattacharjee and Krishnaji (1981) consider, the gamma and log-gamma distributions to provide an approximation to the landholding data. The gamma distribution has density function given by

$$f_X(x) = \frac{\lambda^{\beta} x^{\beta-1} e^{-\lambda x}}{\Gamma(\beta)},\ x \geq 0,\ \lambda, \beta > 0;$$

where $\Gamma(\beta)$ is the gamma function. The proportional failure rate of the gamma distribution is increasing for every λ and β greater than 0, and thus belongs to the class IPFR. However the gamma has the DFR property (decreasing failure rate) for $\beta < 1$ and the IFR property (increasing failure rate) for $\beta > 1$. For $\beta = 1$ we get the exponential distribution which has a constant failure rate. The proportional reversed failure rate is a decreasing function so the gamma distribution is DRPFR.

The log-gamma distribution has a density function given by

$$f_X(x) = \frac{\nu^{\beta} x_0^{\nu} (\log(x/x_0))^{\beta-1}}{x^{\nu+1}\Gamma(\beta)},\ x \geq x_0,\ x_0, \beta, \nu > 0;$$

so that $\log(X/x_0)$ has a gamma distribution with parameters (β, ν). The proportional failure rate for the log-gamma distribution is given by (Bhattacharjee and Krishnaji (1981))

$$r(\log(x/x_0)),\ x > x_0;$$

where $r(x)$ is the failure rate of a gamma distribution with parameters (β, ν). From this relation, it follows that the log-gamma is IPFR for $\beta > 1$ and has a decreasing proportional failure rate $\beta < 1$. For $\beta = 1$ we get the Pareto distribution which has a constant proportional failure rate. Similar computations give us that the log-gamma distribution is DRPFR for $\beta < 1$ and has a decreasing reversed failure rate for $x > x_0 \exp(\frac{\beta-1}{\gamma})$.

DEFINITION 8. For a non negative random variable X, the mean right proportional residual income (MRPRI) is the function

$$e_{X_r''}(\omega) = E\left[X_r''(\omega)\right] = E\left[\frac{X}{\omega}\middle| X \leq \omega\right].$$

By (1) we obtain the following relation between the income gap ratio and the MRPRI

$$\beta(\omega) = 1 - e_{X_r''}(\omega). \qquad \ldots(9)$$

In terms of the MRPRI we define next the following class.

DEFINITION 9. A non negative random variable X is said to be decreasing mean right proportional residual income (DMRPRI) if the mean right proportional residual income is decreasing.

From (9), it is clear that

$$X \text{ is DMRPRI} \Longleftrightarrow \beta(\omega) \text{ is increasing in } \omega.$$

From the characterization of the stochastic ordering (6), we obtain the following result:

PROPOSITION 10. *Let X be a non negative random variable then*

$$X \text{ is DRPFR} \Longrightarrow X \text{ is DMRPRI}.$$

As in the left truncation case, we can characterize the previous classes, in terms of star shaped and Lorenz ordering of right residual incomes. We obtain next the following result.

THEOREM 11. *For a continuous, non negative random variable X*

a) X is DRPFR $\Longleftrightarrow X_r(i) <_* X_r(j)$, $\forall i < j$.

b) X is DMRPRI $\Longleftrightarrow X_r(i) <_L X_r(j)$, $\forall i < j$.

PROOF. a) "$\Longleftarrow$ part "

Let us suppose that $X_r(i) <_* X_r(j)$, $\forall i < j$; i.e.

$$\frac{F^{-1}_{X_r(j)}(q)}{F^{-1}_{X_r(j)}(p)} \geq \frac{F^{-1}_{X_r(i)}(q)}{F^{-1}_{X_r(i)}(p)}, \ \forall 0 < p < q < 1, \ \forall i < j.$$

Observing that $F^{-1}_{X''_r(i)}(p) = F^{-1}_{X_r(i)}(p)/i$; we can write the previous relation as

$$F^{-1}_{X''_r(j)}(q)F^{-1}_{X''_r(i)}(p) \geq F^{-1}_{X''_r(i)}(q)F^{-1}_{X''_r(j)}(p);$$

and noting that $\lim_{q\to 1^-} F^{-1}_{X''_r(i)}(q) = 1$, we get from the last expression that

$$F^{-1}_{X''_r(i)}(p) \geq F^{-1}_{X''_r(j)}(p),$$

and hence X is DRPFR.

"$\Longrightarrow$ part"

If we denote by $x^p_r(i) = F^{-1}_{X_r(i)}(p)$, then we have that

$$F_X(x^p_r(i)) = pF_X(i).$$

Therefore taking $x^p_r(i)/x^q_r(i)$, $x^p_r(j)/x^q_r(j) \geq 1$ and $x^q_r(j) \geq x^q_r(i)$; we can obtain the following equality

$$F_{X''_r(x^q_r(i))}(x^p_r(i)/x^q_r(i)) = \frac{p}{q} = F_{X''_r(x^q_r(j))}(x^p_r(j)/x^q_r(j));$$

and by the stochastic monotonicity of $X''_r(\alpha)$, we get that

$$F_{X''_r(x^q_r(i))}(x^p_r(i)/x^q_r(i)) \geq F_{X''_r(x^q_r(i))}(x^p_r(j)/x^q_r(j));$$

so that

$$x^p_r(i)/x^q_r(i) \geq x^p_r(j)/x^q_r(j).$$

b) We will use the characterization given in (8) of the Lorenz ordering in terms of the ST2 ordering.

"$\Longleftarrow$ part"

If we denote by $H(x,i) = \int_x^{+\infty} \overline{F}_{X_r(i)/E[X_r(i)]}(t)dt$; then by definition we get that

$$H(x,i) \leq H(x,j),\ \forall x,\ \forall i < j.$$

The function $H(x,i)$ is a differentiable, decreasing and convex function, with $\left.\frac{\partial H(x,i)}{\partial x}\right|_{x=0} = -1$, with the following properties

$$i)\ H(x,i) = 1 - x,\ \forall x \leq 0$$

$$ii)\ H(x,i) > 1 - x,\ \forall x > 0$$

and

$$iii)\ H(x,i) = 0,\ \forall x > i/E\left[X_r(i)\right].$$

Let us suppose that the result does not hold, so that there exists $i < j$, with

$$\frac{j}{E\left[X_r(j)\right]} < \frac{i}{E\left[X_r(i)\right]}.$$

Let us take $j/E[X_r(j)] < x < i/E[X_r(i)]$, we get by i) that

$$0 < H(x,i) \leq H(x,j) = 0;$$

which is a contradiction, therefore X is DMRPRI.

"$\Longrightarrow$ part"

Let us suppose that X is DMRPRI, i.e.

$$\frac{i}{E[X_r(i)]} \leq \frac{j}{E[X_r(j)]}, \ \forall i < j.$$

If $x \geq j/E[X_r(j)] \geq i/E[X_r(i)]$ or $j/E[X_r(j)] \geq x \geq i/E[X_r(i)]$, then it follows easily that $H(x,i) \leq H(x,j)$. Let us suppose that $j/E[X_r(j)] \geq i/E[X_r(i)] > x$, then we get

$$\frac{E[X_r(xE[X_r(i)])]}{xE[X_r(i)]} \geq \frac{E[X_r(xE[X_r(j)])]}{xE[X_r(j)]};$$

and this condition is equivalent to

$$\frac{H(x,i)-(1-x)}{H(x,j)-(1-x)} \text{ is increasing in } x,$$

for $x \leq \frac{i}{E[X_r(i)]}$, therefore

$$\frac{H(x,i)-(1-x)}{H(x,j)-(1-x)} \leq \frac{H(i/E[X_r(i)],i)-(1-i/E[X_r(i)])}{H(i/E[X_r(i)],j)-(1-i/E[X_r(i)])} \leq 1,$$

so $H(x,i) \leq H(x,j)$. □

REMARK. The result a) of this theorem generalizes the theorem 12 given by Belzunce, Candel and Ruiz (1995) for the absolutely continuous case.

Noting that for the power distribution model ($p(\alpha,\beta)$), with distribution function given by

$$F_p(x,\alpha,\beta) = P[p(\alpha,\beta) \leq x] = \left(\frac{x}{\beta}\right)^{\alpha}, \ \ 0 < x < \beta, \ \alpha > 0; \qquad \ldots(10)$$

the RPFR and the MRPRI are constant we get the following corollary from the previous theorem:

COROLLARY 12. *Let X be a continuous, non negative random variable. The following conditions are equivalent:*

a) X follows the power distribution model.

b) $X_r(i) =_* X_r(j)$, $\forall i < j$.

c) $X_r(i) =_L X_r(j)$, $\forall i < j$.

The equivalence between a) and c) was given by Moothathu (1986).

From the previous result we get the following observations. We have observed before that the empirical income distributions belong to the DRPFR class. Hence the right residual incomes presents a decreasing concentration for a decreasing poverty line, and thus the specification of the poverty line is directly related to the inequality among the poor people. By the proposition 10 and (9) we get that the income gap ratio is increasing in ω, and by b) of the previous theorem this implies that all the measures of inequality that preserve the Lorenz ordering, are also increasing in the poverty line. For applications to the monotonicity of left and right truncated measures of inequality see Belzunce, Candel and Ruiz (1995).

When the affluence is studied, similar results are obtained by observing that for $Y = 1/X$ then

$$X \text{ is IPFR} \Longleftrightarrow Y \text{ is DRPFR};$$

$$\beta^*(\theta) = 1 - \frac{E\left[Y_r(\theta')\right]}{\theta'}$$

and

$$G^*(\theta) = 1 - 2\int_0^1 L_{Y_r(\theta')}(p)dp;$$

where $\theta' = 1/\theta$.

3. Asymptotic Behaviour of Residual Incomes

For income distributions, empirical evidence shows that the Pareto distribution is the model for high income groups (Mandelbrot (1960)). In this section we study the previous property in the classes considered in section 2. We observe that in the classes IPFR and DMLPRI, the left residual incomes $X_l(\theta)$, behave like the Pareto distribution, for high values of θ. The distribution function of the Pareto distribution ($P(\alpha, \theta)$), is given by

$$F_P(x, \alpha, \theta) = 1 - \left(\frac{\theta}{x}\right)^{\alpha}\ ;\ x \geq \theta > 0,\ \alpha > 0.$$

Clearly $X_l(\theta)$ behaves like $P(\alpha, \theta)$ if $X_l''(\theta)$ behaves like $P(\alpha, 1)$.

DEFINITION 13. Let X be a continuous, non negative random variable, with distribution F_X. F_X is said to belong to the domain of attraction of left residual incomes of the Pareto distribution ($F_X \in D_{LRI}(P_\alpha)$), if

$$\lim_{\theta \to +\infty} F_{X_l''(\theta)}(x) = F_P(x, \alpha, 1).$$

From this definition it follows that asymptotically the left proportional residual incomes behave like the Pareto, $P(\alpha, 1)$. The previous property has been characterized by Balkema and de Haan (1974) as

$$F_X \in D_{LRI}(P_\alpha) \Longleftrightarrow \quad E\left[X^\xi\right] \text{ and } \lim_{\theta\to+\infty} E\left[(X''_\theta)^\xi\right] \text{ exist and are finite, for } 0<\xi<\alpha. \qquad \ldots(11)$$

We next define the following more general class than the IPFR distribution.

DEFINITION 14. A non negative random variable X is said to be AIPFR (asymptotically increasing proportional failure rate) if there exists a θ_0 such that

$$X''_l(\theta) >_{ST} X''_l(\theta'),\ \forall\theta' > \theta > \theta_0.$$

We then have the following.

THEOREM 15. *Let X be a continuous, non negative random variable, such that $E\left[X^\xi\right]$ is finite with $\xi > 0$, then given $\alpha > \xi$ we have*

$$X \text{ is } AIPFR \Longrightarrow F_X \in D_{LRI}(P_\alpha).$$

PROOF. Firstly, if X is AIPFR we get that there exists an ω_0, such that for every $\theta' > \theta > \theta_0$, $X''_l(\theta) >_{ST} X''_l(\theta')$. By (6) it follows that for every increasing function φ,

$$E\left[\varphi(X''_l(\theta))\right] \geq E\left[\varphi(X''_l(\theta'))\right].$$

So $E\left[\varphi(X''_l(\theta))\right]$ is a decreasing and bounded below function of θ. Therefore if $E\left[\varphi(X''_l(\theta))\right]$ exists for $\varphi(x) = x^\xi$, then the limit $\lim_{\theta\to+\infty} E\left[\varphi(X''_l(\theta))\right]$ exits, and the result follows by (11). The existence of $E\left[\varphi(X''_l(\theta))\right]$ is assured by the existence of $E\left[X^\xi\right]$. □

For the right residual incomes we state next similar results in terms of the power distribution (10).

In order to state the main theorem we need to give the following definitions.

DEFINITION 16. Let X be a continuous, non negative random variable, with distribution F_X, then we say that F_X belongs to the domain of attraction of right residual incomes of the power distribution function $p(\alpha,1)$ ($F_X \in D_{RRI}(p_\alpha)$) if

$$\lim_{\omega\to0+} F_{X''_r(\omega)}(x) = F_p(x,\alpha,1).$$

DEFINITION 17. A non negative random variable X is said to be initially decreasing reversed proportional failure rate (IDRPFR) r.v. if there exists an ω_0, such that

$$X''_r(\omega) >_{ST} X''_r(\omega'),\ \forall\omega < \omega' < \omega_0.$$

We state next the following theorem.

THEOREM 18. *Let X be a continuous, non negative random variable, such that $E[X^{-\xi}]$ exists with $\xi > 0$, then given $\alpha > \xi$, we have*

$$X \text{ is } IDRPFR \Longrightarrow F_X \in D_{RRI}(p_\alpha)).$$

PROOF. The proof follows by considering $Y = 1/X$ and applying the theorem 15. □

Among the models which provide a better fit to the whole income distribution, there are the Singh-Maddala model and the Dagum Model Type I (Dagum (1980) and (1983)). As we have seen the Singh-Maddala and the Dagum model type I are IPFR and DRPFR respectively. Also, it is easy to see that the Singh-Maddala is DRPFR and the Dagum model Type I is IPFR. By the previous theorems we only need to ensure the existence of moments for these models to see that thus belong to $D_{LRI}(P_\alpha)$ and $D_{RRI}(p_\gamma)$. For the Singh-Maddala model the moments $E[X^\xi]$ exist for $\xi < bc$ and $E[X^{-\xi}]$ exists for $\xi < b$. Also, for the Dagum model Type I, $E[X^\xi]$ exist for $\xi < \delta$ and $E[X^{-\xi}]$ exists for $\xi < \beta\delta$. We consider now the models fitted to the U.S. family income distribution of 1978 (Dagum (1983)). For the Singh-Maddala model the estimated parameters are $a = 0.02511$, $b = 1.6945$ and $c = 10.973$. For the Dagum model Type I, the parameters are $\delta = 3.168216$, $\beta = 0.40269$ and $\lambda = 35.1774$. We can see in fig 1, how the right proportional residual incomes of the Singh-Maddala model approximate to the power distribution with $\gamma = b = 1.6945$, for decreasing values of ω. In fig. 2 we observe the same behaviour in this case for $\gamma = \beta\delta = 1.4570$, for the Dagum model Type I.

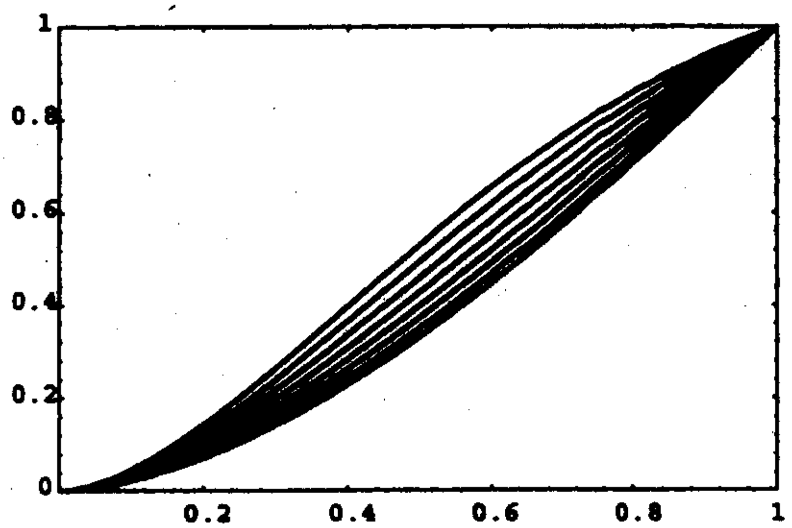

Figure 1. Covergence to the power model $p(1.6945, 1)$, for right proportional residual incomes of the Singh-Maddala model

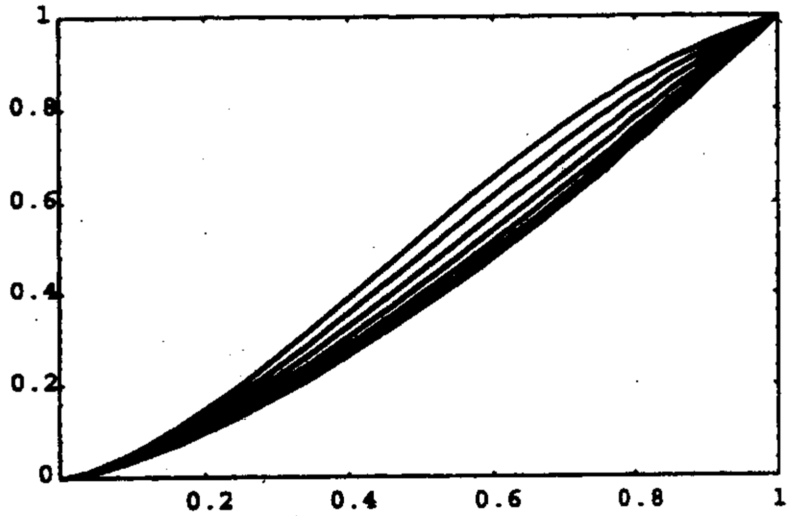

Figure 2. Covergence to the power model $p(1.4570, 1)$, for right proportional residual incomes of the Dagum Model Type 1

4. Estimation of Inequality in the Power Model

As we have seen in the previous section, for low values the right residual income follows, asymptotically, the power distribution. Therefore in the study of poverty it is interesting to consider the estimation of the Lorenz curve and the Gini index for this model. Iyengar (1960) shows that the maximum likelihood estimator of the Gini index for the normal-logarithmic model is asymptotically normal and Moothathu ((1985a), (1985b)) obtains the distribution of the maximum likelihood estimators of the Lorenz curve and the Gini index of the Pareto and exponential distribution, respectively. He also obtains uniformly minimum variance unbiased estimators which are strongly consistent and asymptotically normal, for the lognormal distribution (Moothathu (1989)) and the Pareto distribution (Moothathu (1990)). In this section we derive the maximum likelihood estimators of the Lorenz curve and the Gini index and obtain the exact distribution and asymptotic properties.

For the power distribution model $p(\alpha, \beta)$, the Lorenz curve is given by

$$L_X(p) = p^{1+\frac{1}{\alpha}}, \qquad \ldots(12)$$

and the Gini index by

$$G = \frac{1}{1+2\alpha}. \qquad \ldots(13)$$

If we consider a random sample of size n, $(X_1, X_2, ..., X_n)$, it is easy to see that the MLE of the parameter $\theta = (\alpha, \beta)$ is given by $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$, where

$$\hat{\alpha} = \frac{n}{\sum_{j=1}^{n}(-\ln X_j + \ln X_{(n)})}, \qquad \hat{\beta} = X_{(n)},$$

where $X_{(n)} = \max(X_1, X_2, ..., X_n)$.

The random variable $Y_i = -\ln X_i$ has the following pdf

$$g(y) = \alpha\ e^{-\alpha(y - \ln\beta)}, \quad \ln\beta < y < +\infty,$$

and thus follows an exponential distribution. Noting that $\ln X_{(n)} = -Y_{(1)}$, where $Y_{(1)} = \min(Y_1, Y_2, ..., Y_n)$, the MLE of $\hat{\alpha}$ can be rewritten in terms of the random sample $(Y_1, Y_2, ..., Y_n)$ as follows

$$\hat{\alpha} = \frac{n}{\sum_{j=1}^{n}(Y_j - Y_{(1)})}.$$

If we denote by

$$S = \frac{1}{n}\sum_{j=1}^{n}(Y_j - Y_{(1)}),$$

applying the Zehna's theorem, the MLEs of

$$\lambda_1(\alpha) = \frac{1}{\alpha} \quad \text{and} \quad \lambda_2(\alpha) = \frac{1}{1+2\alpha}$$

are given, respectively, by

$$\hat{\lambda}_1 = S \quad \text{and} \quad \hat{\lambda}_2 = \frac{S}{2+S}. \qquad \ldots(14)$$

In order to obtain the exact distribution of the MLE of the Lorenz curve for a fixed p, we need to obtain the distribution of $\hat{\lambda}_1$. We will make use of the following result of Sukhatme (1937) who showed that the random variable $Z = n\alpha S$ follows the gamma distribution $\gamma(1, n-1)$, with pdf

$$k(z) = \frac{1}{\Gamma(n-1)} z^{n-2}\ e^{-z}, \quad z > 0.$$

Therefore the pdf $h_1(s)$, of S is given by

$$h_1(s) = \frac{(n\alpha)^{n-1}}{\Gamma(n-1)} s^{n-2}\ e^{-n\alpha s}, \quad s > 0,$$

which shows that $\hat{\lambda}_1$ follows a gamma distribution $\gamma(n\alpha, n-1)$.

From (14) it follows that

$$Z = \frac{2n\alpha\hat{\lambda}_2}{1-\hat{\lambda}_2},$$

and so $\hat{\lambda}_2$ has pdf

$$h_2(t) = \frac{(2n\alpha)^{n-1}}{\Gamma(n-1)} \frac{t^{n-2}}{(1-t)^n} e^{-\frac{2n\alpha t}{1-t}}, \quad 0 < t < 1.$$

Further, for $r > 0$,

$$E(\hat{\lambda}_1^r) = \frac{\Gamma(n+r-1)}{(n\alpha)^r \, \Gamma(n-1)}.$$

and

$$E(\hat{\lambda}_2^r) = \frac{(2n\alpha)^{n-1}}{\Gamma(n-1)} \int_0^{+\infty} t^{n+r-2} \, (1+t)^{-r} \, e^{-2n\alpha t} \, dt.$$

If we consider the confluent hypergeometric function $U(a,b,c)$, which is given by (Abramowitz and Stegun (1968))

$$U(a,b,c) = \frac{1}{\Gamma(a)} \int_0^{+\infty} t^{a-1} \, (1+t)^{b-a-1} \, e^{-ct} \, dt,$$

we get

$$E(\hat{\lambda}_2^r) = \frac{(2n\alpha)^{n-1}\Gamma(n+r-1)}{\Gamma(n-1)} \, U(n+r-1, n, 2n\alpha).$$

For the asymptotic properties of the previous estimators we observe first that S can be expressed as

$$S = \frac{n-1}{n} \frac{\sum_{j=2}^{n} T_j}{n-1};$$

where the T_j, are iid exponential with mean $1/\alpha$. By applying SLLN to S we get that $S \xrightarrow{a.s} 1/\alpha$, and thus we get that

$$\widehat{\lambda}_1 \xrightarrow{a.s} \lambda_1 \quad \text{and} \quad \widehat{\lambda}_2 \xrightarrow{a.s} \lambda_2.$$

We get also that

$$\frac{\sqrt{n-1}}{\lambda_1} \left(\frac{n}{n-1} \widehat{\lambda}_1 - \lambda_1 \right)$$

converges in distribution to the standard normal distribution. This random variable can be expressed as

$$\frac{\sqrt{n-1}}{\lambda_1} \left(\frac{n}{n-1} \widehat{\lambda}_1 - \lambda_1 \right) = \sqrt{n-1} \left(\frac{\widehat{\lambda}_1}{\lambda_1} - 1 \right) + \frac{\sqrt{n-1}}{n-1} \frac{\widehat{\lambda}_1}{\lambda_1},$$

and by Slutsky's theorem we get that $\sqrt{n-1}\left(\widehat{\lambda}_1/\lambda_1 - 1\right)$ converges in distribution to the standard normal distribution. From the relations between $\widehat{\lambda}_2$ and λ_2 with $\widehat{\lambda}_1$ and λ_1 respectively, we get that

$$\frac{\sqrt{n-1}}{\lambda_2(1-\lambda_2)}\left(\widehat{\lambda}_2 - \lambda_2\right)$$

converges in distribution to the standard normal distribution.

From (12) we get that the MLE of the Lorenz curve for a fixed p is given by

$$\hat{L}_X(p) = p^{1+\hat{\lambda}_1}, \quad p \in (0,1),$$

and from (13) the MLE of the Gini index is given by

$$\widehat{G} = \hat{\lambda}_2.$$

By the results given previously we can get easily the properties of $\hat{L}_X(p)$ and $\widehat{G}$.

Acknowledgements. The authors are pleased to thank the referees, whose useful comments have improved the presentation of this paper.

References

Abramowitz, M. and Stegun, I.A. (1968). *Handbook of Mathematical Functions.* Dover Publications.

Arnold, B.C. (1987). *Majorization and the Larenz Order: A Brief Introduction,* Lectures Notes in Statistics **43**. Springer-Verlag.

Balkema, A.A. and de Haan, L. (1974). Residual life time at great age. *The Annals of Probability,* **2**, 792-804.

Barlow, R.E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing.* Holt, Rinehart and Winston, Inc.

Belzunce, F., Candel, J. and Ruiz, J.M. (1995). Ordering of truncated distributions through concentration curves. *Sankhyā*, Ser. A, 57, 375-383.

Bhattacharjee, M.C. (1988). Reliability ideas and applications in economics and social sciences in *Handbook of Statistics, Vol. 7: Quality Control and Reliability,* P.R. Krishnaiah and C.R. Rao (eds.), 175-213. North-Holland, Amsterdam.

— — —– (1993). How rich are the rich? Modeling affluence and inequality via reliability theory. *Sankhyā*, Ser. B, **55**, 1-26.

Bhattacharjee, M.C. and Krishnaji, N. (1981). DFR and other heavy tail properties in modelling the distribution of land and some alternative measures of inequality in *Statistics: Applications and New Directions,* J.K. Ghosh (ed.), 100-115. Indian Statistical Institute, Calcutta.

Bryson, M.C. and Siddiqui, M.M. (1969). Some criteria for ageing. *Jour. Amer. Statist. Assoc.*, **64**, 1472-1483.

Chandra, M. and Singpurwalla, N.D. (1981). Relationships between some notions which are common to reliability theory and economics. *Mathematics of Operations Research,* **6**, 113-121.

Dagum, C. (1977). A new model of personal income distribution: specification and estimation. *Economie Appliquée*, **30**, 413-437.
— — — (1980). The generation and distribution of income, the lorenz curve and the gini ratio. *Economie Appliquée*, **33**, 327-367.
— — — (1983). Income distribution models in *Encyclopedia of Statistical Sciences*, Vol. **4**, S. Kotz *et al. (eds)*, Wiley-Interscience, 27-34.
Iyengar, N.S. (1960). On the standard error of the lorenz concentration ratio. *Sankhyā*, **22**, 371-378.
Kakwani, N.C. (1980). *Income Inequality and Poverty, Methods of Estimation and Policy Applications*. Oxford University Press, New-York.
Klefsjö, B. (1984). Reliability interpretations of some concepts from economics. *Naval Research Logistics Quarterly*, **31**, 301-308.
Kochar, S.C. (1989). On extensions of DMRL and related partial orderings of life distributions. *Commun. Statist.- Stochastic Models*, **5**, 235-245.
Mandelbrot, B. (1960). The Pareto-Lévy law and the distribution of income. *International Economic Review*, **1**, 79-106.
Marshall, A.W. and Olkin, I. (1979). *Inequalities: Theory of Majorization and Its Applications*. Academic Press, Inc.
Moothathu, T.S.K. (1985a). Sampling distributions of lorenz curve and gini index of the pareto distribution. *Sankhyā*, Ser. B, **47**, 247-258.
— — — (1985b). Distributions of maximum likelihood estimators of the Lorenz curve and Gini index of exponential distribution. *Ann. Inst. Statist. Math.*, **37**, 473-479.
— — — (1986). A characterization of power function distribution through a property of the Lorenz curve. *Sankhyā*, Ser. B, **48**, 262-265.
— — — (1989). On unbiased estimation of Gini index and Yntema-pietra index of lognormal distribution and their variances. *Commun. Statist. Theory Meth.*, **18**, 661-672.
— — — (1990). The best estimator and strongly consistent asymptotically normal unbiased estimator of Lorenz curve Gini index and theil entropy index of pareto distribution. *Sankhyā*, Ser. B, **52**, 115-127.
Ord, J.K.; Patil, G.P. and Taillie, C. (1983). Truncated distributions and measures of income inequality. *Shankyā*, Ser. B, **45**, 413-430.
Pham, T.G. and Turkkan, N. (1994). The lorenz curve and the scaled total-time-on-test transform curves: A Unified Approach. *IEEE Transactions on Reliability*, **43**, 76-84.
Rolski, T. (1976). Order relations in the set of probability distributions functions. *Dissertationes Mathematicae*, no. **132**.
Sen, A.K. (1976). The measurement of poverty: an axiomatic approach. *Econometrica*, **44**, 219-232.
Sen, P.K. (1986). The Gini coefficient and poverty indexes: some reconciliations *Jour. Amer. Statist. Assoc.*, **81**, 1050-1057.
— — — (1988). The harmonic Gini coefficient and affluence indexes. *Mathematical Social Sciences*, **16**, 65-76.
Singh, S.K. and Maddala, G.S. (1976). A function for size distribution of incomes. *Econometrica*, **44**, 963-970.
Sukhatme, P.V. (1937). Tests of significance for samples of the χ^2 population with two degrees of freedom. *Ann. Eugen.*, **8**, 52-56.
Taillie, C. (1981). Lorenz ordering within the generalized gamma family of income distributions in *Statistical Distributions in Scientific Work*, Vol. **6**, C. Taillie et al. (eds.), 181-192. D. Reidel Publishing Company.
Takayama, N. (1979). Poverty, income inequality and their measures: professor Sen's axiomatic approach reconsidered. *Econometrica*, **47**, 747-760.

F. Belzunce, J. Candel and J.M. Ruiz
Departamento de Estadística e Investigación Operativa
Facultad de Matemáticas. Universidad de Murcia
Campus de Espinardo
30100 Espinardo (Murcia)
SPAIN
e-mail : belzunce@fcu.um.es, jcandel@fcu.um.es, jmruizgo@fcu.um.es