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CHARACTERIZATION THROUGH MOMENTS OF THE RESIDUAL LIFE AND CONDITIONAL SPACINGS*

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SUMMARY. In this work, we give a general method to obtain a distribution function $F(x)$ through the moment of the residual life defined by $h_k(x) = E((X-x)^k \mid X \geq x)$, for $k = 1, 2, 3, ...$, both in continuous and discrete cases. We also characterize $F(x)$ through moments of conditional spacings of order statistics, which have applications in the context of the k-out-of-n systems. Moreover, we study characterizations based on relations between failure rate function and left censured moment functions, $m_k(x) = E(X^k \mid X \geq x)$.

1. Introduction

Let X be a random variable (r.v.), usually representing the life length for a certain unit (where this unit can have multiple interpretations), then r.v. $(X - x \mid X \geq x)$, represents the residual life of a unit with age x.

Several functions are defined related to the residual life. The failure rate function, defined by:

$$r(x) = \frac{f(x)}{1 - F(x-)} \qquad \ldots (1.1)$$

represents the failure rate of X (or F) at age x, for $x \in D = \{t \in \mathcal{R} : \mathcal{F}(\sqcup-) < \infty\}$, where $F(x) = P(X \leq x)$, $F(x-) = \lim_{z \to x^-} F(z)$ and $f(x)$ is the density function when X is absolutely continuous, or $f(x) = P(X = x)$ when X is discrete.

Another interesting function is the mean residual life function, defined by $h_1(x) = E(X - x \mid X \geq x)$, for $x \in D$, and it represents the expected additional

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life length for a unit which is alive at age x. This function is equivalent to the left censured mean function, also called vitality function (see Gupta (1975)), defined by $m_1(x) = E(X \mid X \geq x)$.

It is well known that these three functions uniquely determine $F(x)$ (see Gupta (1975), Kotz and Shanbhag (1980) and Zoroa *et al.* (1990)). In particular, the explicit expression of $F(x)$ from $h_1(x)$ (inversion formula), is given by

$$F(x) = 1 - \exp\left\{ - \int_{-\infty}^{x} \frac{h_1'(t)+1}{h_1(t)} dt \right\} \qquad \ldots (1.2)$$

for $x \in D$ and absolutely continuous F.

Other important functions which also uniquely determine $F(x)$ are the right censured mean function,$m_1^*(x) = E(X \mid X \leq x)$, the doubly censored mean function,$m_1^{**}(x,y) = E(X \mid x \leq X \leq y)$, and the order mean function, $\xi_i(x) = E(X_{i+1,n} \mid X_{i,n} = x)$, where $X_{i,n}$ denotes the ith order statistics of a random sample of size n from F (see Ruiz and Navarro (1995, 1996) and Franco and Ruiz (1995)).

Naturally, some authors have recently studied characterization problems through left censured moments $m_k(x) = E(X^k \mid X \geq x)$ (see Gupta and Gupta (1983), Adatia *et al.* (1991) and Ghitany *et al.* (1995)) and moments of the residual life, $h_k(x) = E((X-x)^k \mid X \geq x)$ (see Nagaraja (1975), Dallas (1979), Gupta and Gupta (1983) and Galambos and Hagwood (1992)), which generalize left censured mean function and mean residual life, respectively.

For $m_k(x)$, it is easy to obtain the equality

$$r(x) = \frac{m_k'(x)}{m_k(x) - x^k} \qquad \ldots (1.2)$$

for $k = 1, 2, \ldots$, which proves that $m_k(x)$ uniquely determines $F(x)$.

Nagaraja (1975) shows that, under some conditions, $h_k(x)$, uniquely determines $F(x)$ in the absolutely continuous case. Dallas (1979) characterizes the exponential distribution when $h_k(x)$ is constant. Later, Gupta and Gupta (1983) prove that $h_2(x)$ does not uniquely determine $F(x)$. Recently, Galambos and Hagwood (1992) have shown for positive and continuous random variables that $h_2(x)$ uniquely determines $F(x)$, and hence, that the result of Gupta and Gupta has an error. Nevertheless, neither Nagaraja (1975) nor Galambos and Hagwood (1992) give an explicit expression to obtain $F(x)$ from $h_2(x)$.

In Section 2, we give a general method to obtain $F(x)$ from $h_k(x)$ for $k = 1, 2, 3, \ldots$, both in continuous and discrete cases. In particular, when $k = 1$, we obtain the results of Laurent (1974), Gupta (1975) and Zoroa et al. (1990) for the mean residual life.

In Section 3, we study characterization problems through moments of conditional spacings

$$\xi_{i,k}(x) = E((X_{i+1,n} - X_{i,n})^k \mid X_{i,n} = x) \qquad \ldots (1.4)$$

which represents the kth moment of the residual life of a $(n-i)$-out-of-n system, knowing that only $(n-i)$ components are alive at age x, from this one deduces its importance on system reliability. So, we prove that $\xi_{i,k}$ uniquely determines F. In particular, when $k=1$, we obtain the results given by Ferguson (1967), Kirmani and Alam (1980), Nagaraja (1988a, 1988b) and Franco and Ruiz (1995).

Another important problem is to characterize distribution functions from relations between reliability functions. For example, Ruiz and Navarro (1994) give a general way to characterize a distribution function through relation of the type

$$m_1(x) = c + d(x)r(x) \qquad \ldots(1.5)$$

where c is a constant, and $d(x)$ is a real function satisfying some conditions. In particular, taking $c=0$, it is shown that $m_1(x)/r(x)$ uniquely determines $F(x)$. The work of Ruiz and Navarro (1994), extends previous results given by different authors (see Osaki and Li (1988), Ahmed (1991) and Nair and Sankaran (1991)).

Recently, particular characterizations have been given through relations of type

$$m_k(x) = c + d(x)r(x) \qquad \ldots(1.6)$$

where $k = 2, 3...$ (see Adatia *et al.* (1991), Koicheva (1993) and Ghitany et al. (1995)).

In Section 4, we give a general method to obtain $F(x)$ from a relation of type (1.6), both in continuous and discrete cases, extending results given in Adatia *et al.* (1991), Koicheva (1993) and Ghitany *et al.* (1995).

It is easy to translate our results to the corresponding concepts for the right or doubly censored random variable.

2. Characterization Through Moments of the Residual Life.

In this section we study determination of $F(x)$, from the moments of the residual life $h_k(x)$, defined by

$$h_k(x) = \frac{1}{1-F(x-)}\int_x^{\infty}(t-x)^k dF(t) \qquad \ldots(2.1)$$

for $k = 1, 2, ...$ and $x \in D = \{x \in \mathcal{R} : \mathcal{F}(\S-) < \infty\}$. From now on we assume that

$$\int_0^{\infty} t^k dF(t) \qquad \ldots(2.2)$$

is finite, which implies the existence of $h_k(x)$. Let us see some previous results.

LEMMA 2.1. *If X is an absolutely continuous r.v., then*

$$r(x) = \frac{h_k'(x) + kh_{k-1}(x)}{h_k(x)} \qquad \ldots(2.3)$$

holds, for all $x \in D$ *and for all* $k = 1, 2, 3, ...$

REMARK 2.1. The equality (2.3) proves that two consecutive moments of the residual life determine $F(x)$.

REMARK 2.2. Using (2.3) for k and $k+1$, and taking $\lambda_k = h_{k+1}/h_k$, we have

$$\lambda_{k-1}(x) = \frac{k\lambda_k(x)}{\lambda'_k(x) + k + 1} \qquad \ldots (2.4)$$

which proves that the ratio of two consecutive moments of the residual life, uniquely determines $F(x)$.

Another interesting concepts given by Gupta and Gupta (1983), are the partial moments, defined by

$$g_k(x) = \int_x^{\infty} (t-x)^k dF(t) \qquad \ldots (2.5)$$

for $k = 1, 2, ...$ Some applications of partial moments in Bayesian point estimation and in management science problems have been mentioned by Winkler *et al.* (1972). Gupta and Gupta (1983) show that $g_k(x)$ uniquely determines $F(x)$. However, they do not give the explicit expression of $F(x)$ from $g_k(x)$, which we obtain in the following proposition.

PROPOSITION 2.2. If F is an absolutely continuous (life) d.f., then

$$F(x) = 1 - \frac{(-1)^k}{k!} g_k^{(k)}(x) \qquad \ldots (2.6)$$

holds for all $k = 1, 2, ...$

In the following theorem we obtain $F(x)$ from $h_k(x)$.

THEOREM 2.3. *Let* $F(x)$ *be a distribution function with support* (α, β), $\alpha \in \mathcal{R}$, β *finite or infinite, and analytic in* $[\alpha, \beta)$ *with*

$$\frac{(-1)^k k!}{h_k(x)} = \sum_{i=0}^{\infty} b_i (x-\alpha)^i, \qquad \ldots (2.7)$$

then $h_k(x)$ *uniquely determines* $g_k(x) = \sum_{i=0}^{\infty} a_i (x-\alpha)^i$, *and hence,* $F(x)$, *through*

$$a_j = \begin{cases} \frac{1}{j!} h_k^{(j)}(\alpha-) & j = 0, 1, ..., k-1 \\ \frac{b_0 a_{j-k} + \ldots + b_{j-k} a_0}{j!/(j-k)!} & j = k, k+1, ... \end{cases} \qquad \ldots (2.8)$$

PROOF. Since $F(\alpha-) = 0$, from (2.1) and (2.5), we have

$$h_k(x) = g_k(x) \qquad \ldots (2.9)$$

for all $x < \alpha$, and since g_k has continuous differential of order $k-1$, if we know $h_k(x)$, then we know $g_k^{(j)}(\alpha)$ for $j = 0, 1, ..., k-1$.

Using that $F(x)$ is analytic in $[\alpha, \beta)$ and

$$g_k^{(k)}(x) = (-1)^k k!(1 - F(x)) \qquad \ldots (2.10)$$

we obtain that $g_k(x)$ is also analytic, and we can write

$$g_k(x) = \sum_{i=0}^{\infty} a_i (x - \alpha)^i \qquad \ldots (2.11)$$

for $x \in [\alpha, \beta)$, where

$$a_j = \frac{1}{j!} h_k^{(j)}(\alpha-) \qquad \ldots (2.12)$$

and hence, $h_k(x)$ uniquely determines a_j, for $j = 0, 1, ..., k-1$. ¿From definition of $h_k(x)$ and equality (2.10) it is easy to obtain the following differential equation

$$g_k^{(k)}(x) = (-1)^k \frac{k!}{h_k(x)} g_k(x) \qquad \ldots (2.13)$$

Moreover, $h_k(x)$ is positive and analytic in $[\alpha, \beta)$, so (2.7) and (2.13) lead to

$$\sum_{i=0}^{\infty} \frac{(i+k)!}{i!} a_{k+i} (x - \alpha)^i = \left(\sum_{i=0}^{\infty} b_i (x - \alpha)^i \right) \left(\sum_{i=0}^{\infty} a_i (x - \alpha)^i \right) \qquad \ldots (2.14)$$

and we have

$$a_{k+i} = \frac{b_0 a_i + ... + b_i a_0}{(i+k)!/i!} \qquad \ldots (2.15)$$

for $i = 0, 1, ...$, that is to say, that from b_i, we obtain a_i for $i = k, k+1, ...$ From (2.12) and (2.15) we have that $h_k(x)$ determines $g_k(x)$ and, hence, $F(x)$, which finishes the proof. □

REMARK 2.3. If we know $h_k(x)$ solving the standard differential equation (2.13), we obtain $g_k(x)$ and, hence $F(x)$. In particular, when $k = 1$, solving (2.13), we have the inversion formula (1.2).

REMARK 2.4. Taking $h_k(x) = k!/a^k$ we obtain the characterization given in Dallas (1979) for the exponential distribution.

REMARK 2.5. Taking $h_k(x) = (b - x)^k / \binom{c+k}{k}$ or $h_k(x) = (a + x)^k / \binom{c-1}{k}$ we characterize the power and Pareto distributions, respectively.

We have not found any previous results for discrete distributions. In this case, we can obtain $F(x)$ from $h_k(x)$, as follows:

THEOREM 2.4. *Let $F(x)$ be the distribution function of a discrete r.v. X with mass in $\{x_i : i = a, a+1, ..., b\}$, where $x_i < x_{i+1}$, a can be $-\infty$ and b can* *be ∞, and let $h_k(x) = E((X-x)^k \mid X \geq x)$. Then, $h_k(x)$ uniquely determines $F(x)$ through the following inversion formula*

$$F(x) = 1 - \prod_{x_i \leq x} \frac{h_k(x_i)}{h_k(x_i+)} \qquad \ldots (2.16)$$

valid for all $x < x_b$, where $h_k(x+) = \lim_{t \to x^+} h_k(t)$.

PROOF. $p_i = P(X = x_i)$ and $x \in (x_j, x_{j+1}]$, then

$$h_k(x_j) = \frac{1}{\sum_{i \geq j} p_i} \sum_{i \geq j} (x_i - x_j)^k p_i \qquad \ldots (2.17)$$

$$h_k(x) = \frac{1}{\sum_{i > j} p_i} \sum_{i > j} (x_i - x)^k p_i \qquad \ldots (2.18)$$

and hence

$$h_k(x_j+) = \frac{1}{\sum_{i > j} p_i} \sum_{i \geq j} (x_i - x_j)^k p_i \qquad \ldots (2.19)$$

¿From (2.17) and (2.19),

$$\frac{h_k(x_j)}{h_k(x_j+)} = \frac{1 - F(x_j)}{1 - F(x_j-)} \qquad \ldots (2.20)$$

holds, and it is easy to obtain (2.16). □

REMARK 2.6. In particular, when $k = 1$, we have the inversion formula for the residual life in the discrete case given by Laurent (1974).

REMARK 2.7. Taking $h_k(x) = \alpha_k$, where $\alpha_k = E(X^k)$, we characterize the geometric distribution.

REMARK 2.8. Using Helly's second theorem it is easy to show that if $F_n \to F$ weakly, then $h_{k,n} \to h_k$ weakly, for all k and hence, if $(X_1, ..., X_n)$ is a sample of size n, then the empirical estimator

$$\widehat{h}_k(x) = \frac{\sum_i (X_i - x)^k \mathbf{1}_{(\mathbf{X}_i \geq \mathbf{x})}}{\sum_i \mathbf{1}_{(\mathbf{X}_i \geq \mathbf{x})}}, \qquad \ldots (2.21)$$

converges weakly to $h_k(x)$, where $\mathbf{1}_{(\mathbf{T})} = \mathbf{1}$ if T is true and $\mathbf{1}_{(\mathbf{T})} = 0$ otherwise. This estimator jointly with Theorem 2.3 can be used to make tests for exponentiality in the same way that Hollander and Proschan (1975) and Lai (1994).

3. Characterization Through Moments of Conditional Spacings.

In a similar way as in Section 2, it can be seen that if F is an absolutely continuous distribution then

$$r(x) = \frac{\xi'_{i,k}(x) + k\xi_{i,k-1}(x)}{\xi_{i,k}(x)} \qquad \ldots (3.1)$$

holds, for all i, $1 \leq i < n$, $x \in D$ and for all $k = 1, 2, 3, \ldots$, where $\xi_{i,k}(x)$ is given by (1.4). We also suppose that (2.2) is finite, which implies that $\xi_{i,k}(x)$ exists. So, remarks similar to 2.1 and 2.2 hold in this case.

Using an analogous development to Proposition 2.2 and Theorem 2.2 , we get the following result:

PROPOSITION 3.1. *Let i be fixed with $1 \leq i < n$. If $D = (\alpha, \beta)$ with $\alpha \in \mathcal{R}$ and F is analytic in $[\alpha, \beta)$, then $\xi_{i,k}$ uniquely determines F.*

REMARK 3.1. If $k = 1$ it is easy to obtain the inversion formula for $\xi_{i,1}$. In particular, when $\xi_{i,1}$ is linear, it obtain the results of Ferguson (1967) and Nagaraja (1988a) for the exponential and beta distributions.

REMARK 3.2. Taking $\xi_{i,k}(x) = \alpha_k/(n-i)^k$ or $\xi_{i,k}(x) = (b-x)^k/\binom{c(n-i)+k}{k}$, we characterize the exponential and power distributions, respectively.

On the other hand, for discrete distributions, we consider two cases for (1.4):

$$\xi_{i,k}(x) = E((X_{i+1,n} - X_{i,n})^k \mid X_{i,n} = x, X_{i+1,n} - X_{i,n} \geq 0) \qquad \ldots (3.2)$$

$$\xi_{i,k}(x) = E((X_{i+1,n} - X_{i,n})^k \mid X_{i,n} = x, X_{i+1,n} - X_{i,n} > 0) \qquad \ldots (3.3)$$

since when $X_{i+1,n} - X_{i,n} \geq 0$, and only in this case, ties in the sample values are permitted. So, we observe that there are very few works based on conditional expectations of order statistics in the discrete case, and these frequently use a sample size $n = 2$, probably due to the possibility of ties, since in this situation the order statistics, $X_{i,n}$, fail to form a Markov chain (see Nagaraja (1982)).

For $X_{i+1,n} - X_{i,n} > 0$, (3.3) can be written as

$$\xi_{i,k}(x) = \sum_{x_j > x} (x_j - x)^k \frac{(1 - F(x_j-))^{n-i} - (1 - F(x_j))^{n-i}}{(1 - F(x))^{n-i}} \qquad \ldots (3.4)$$

and using a similar development to Theorem 2.4, we obtain the following result.

PROPOSITION 3.2. *Let i be fixed with $1 \leq i < n$. If F is a discrete distribution with mass in $\{x_i : i = a, a+1, \ldots, b\}$, where $x_i < x_{i+1}$, a can be $-\infty$ and b can be ∞, then $\xi_{i,k}$ given in (3.3) uniquely determines F by*

$$F(x) = 1 - \left(\prod_{x_j \leq x} \frac{\xi_{i,k}(x_j-)}{\xi_{i,k}(x_j)} \right)^{1/(n-i)} \qquad \ldots (3.5)$$

valid for all $x < x_b$.

REMARK 3.3. If $\xi_{i,1}(x)$ is constant for $x = 0, 1, \ldots$ we characterize the geometric distribution. In particular, taking $n = 2$ in (3.5), the linearity of $\xi_{1,1}$ on the support allow us to obtain the results of Kirmani and Alam (1980) and Nagaraja (1988b) for the geometric, hypergeometric and Waring distributions. In general, for $n \geq 2$, taking $\xi_{i,k}(x) = \alpha_{n-i,k}/q^{n-i}$ we characterize the geometric distribution, where $\alpha_{n-i,k}$ is the kth moments of the geometric distributions of parameter $1 - q^{n-i}$.

For $X_{i+1,n} - X_{i,n} \geq 0$, (3.2) can be written as

$$\xi_{i,k}(x) = \sum_{x_j > x} (x_j - x)^k \left((1 - F(x_j-))^{n-i} - (1 - F(x_j))^{n-i}\right) A(x) \quad \ldots(3.6)$$

where

$$A(x) = \frac{(F(x))^i - (F(x-))^i}{i \int_{F(x-)}^{F(x)} t^{i-1}(1-t)^{n-i} dt}. \quad \ldots(3.7)$$

As we have seen, the positive chance of ties between $X_{i,n}$ and $X_{i+1,n}$ has limited the work in this way, so we show in the following proposition that if $i = 1$ then $\xi_{1,k}$ uniquely determines F, which generalizes to $n \geq 2$ the results of Kirmani and Alam (1980) and Nagaraja (1988b) for $n = 2$. For that, let us see now a previous lemma.

LEMMA 3.3. *For all $n \geq 2$ and θ such that $1 < \theta < n$, the polynomial*

$$q(x) = x^{n-1} + x^{n-2} + \ldots + x^2 + x + (1 - \theta)$$

has a unique point $y \in (0, 1)$ such that $q(y) = 0$.

PROPOSITION 3.4. If F is a discrete distribution with mass in $\{x_i : i = a, a+1, \ldots, b\}$, where $x_i < x_{i+1}$, a can be $-\infty$ and b can be ∞, then $\xi_{1,k}$ given in (3.2) uniquely determines F.

PROOF. From (3.6), we obtain for $x = x_j$ that

$$\begin{aligned} &\xi_{1,k}(x_j) \sum_{s=0}^{n-1} (1 - F(x_j-))^{n-1-s}(1 - F(x_j))^s = \\ &= n \sum_{x_m > x_j} (x_m - x_j)^k \left((1 - F(x_m-))^{n-1} - (1 - F(x_m))^{n-1}\right) \end{aligned} \quad \ldots(3.8)$$

and for $x \in (x_{j-1}, x_j)$ that

$$\xi_{1,k}(x)(1 - F(x_j-))^{n-1} = \sum_{x_m > x} (x_m - x)^k \left((1 - F(x_m-))^{n-1} - (1 - F(x_m))^{n-1}\right) \quad \ldots(3.9)$$

so, from (3.8) and (3.9), we have

$$n\xi_{1,k}(x_j-)(1 - F(x_j-))^{n-1} = \xi_{1,k}(x_j) \sum_{s=0}^{n-1} (1 - F(x_j-))^{n-1-s}(1 - F(x_j))^s$$

which is equivalent to

$$n\frac{\xi_{1,k}(x_j-)}{\xi_{1,k}(x_j)} = \sum_{s=0}^{n-1}\left(\frac{1-F(x_j)}{1-F(x_j-)}\right)^s \qquad \ldots(3.10)$$

taking into account Lemma 3.3 with $\theta = \theta_j = n\frac{\xi_{1,k}(x_j-)}{\xi_{1,k}(x_j)} \in (1,n)$, we have that there exists a unique y_j such that $q(y_j)=0$, and using (3.10) we obtain that

$$y_j = \frac{1-F(x_j)}{1-F(x_j-)},$$

and hence it is easy to get that

$$F(x_j) = 1 - \prod_{x_m \leq x_j} y_m$$

where y_m is uniquely determined by Lemma 3.3 for each $\theta_m = n\frac{\xi_{1,k}(x_m-)}{\xi_{1,k}(x_m)}$. □

REMARK 3.4. From Lemma 3.3 for $n = \{2,3,4,5\}$, the inversion formula for any F can be obtained. For example,

- If $n=2$ then $F(x_j) = 1 - \prod_{x_m \leq x_j}\left(2\frac{\xi_{1,k}(x_m-)}{\xi_{1,k}(x_m)} - 1\right)$.
- If $n=3$ then $F(x_j) = 1 - \prod_{x_m \leq x_j}\left(\left(3\frac{\xi_{1,k}(x_m-)}{\xi_{1,k}(x_m)}\right)^{1/2} - \frac{1}{2}\right)$.

However, the general inversion formula cannot be found for $n > 5$. It is still unknown if discrete distributions will be uniquely determined by moments of conditional spacings, $\xi_{i,k}$, when we impose the condition $i > 1$ in (3.2).

REMARK 3.5. If $\xi_{1,1}$ is constant then the geometric distribution is characterized. In particular, when $n = 2$, we obtain the results of Kirmani and Alam (1980) and Nagaraja (1988b). In general, when $n \geq 2$, taking $\xi_{1,k}(x) = n(1-q)\alpha_{n-1,k}/(1-q^n)$ we characterize the geometric distribution.

4. Characterization Through Relations.

In this section we characterize a distribution function from a relation between the failure rate $r(x)$ and left censured moments $m_k(x)$, both in continuous and discrete case. As in the above sections, we assume that (2.2) is finite, so $m_k(x)$ exists.

THEOREM 4.1. *Let X be a r.v. with differentiable density in its support (α,β). Then, the following conditions are equivalent:*

1. $m_k(x) = c + d(x)r(x)$

2. $f'(x)/f(x) = (c - d'(x) - x^k)/d(x)$

where, c is a constant and $d(x)$ is a real function satisfying

$$\lim_{x\to\beta} f(x)d(x) = 0 \qquad \ldots (4.1)$$

PROOF. Let us go on to prove $(i) \Rightarrow (ii)$. From (i),

$$\int_x^\infty (t^k - c)f(t)dt = d(x)f(x) \qquad \ldots (4.2)$$

holds, and, we obtain

$$-(x^k - c)f(x) = d'(x)f(x) + d(x)f'(x) \qquad \ldots (4.3)$$

which implies (ii).

Reciprocally, rewriting (ii), we have

$$(c - x^k)f(x) = (d(x)f(x))' \qquad \ldots (4.4)$$

and using (4.1), it is easy to obtain (i).

REMARK 4.1. If in the above theorem, $\lim_{x\to\alpha} f(x)d(x) = 0$ holds, then $c = \alpha_k$, where $\alpha_k = E(X^k)$. If we take $c = 0$, the above theorem proves that the ratio $m_k(x)/r(x)$, uniquely determines $F(x)$.

REMARK 4.2. ¿From the above theorem, taking

1. $c = k!a^{-k}$ and $d(x) = \sum_{i=1}^{k} \frac{k!}{i!} a^{i-k-1} x^i$, we obtain the characterizations given in Adatia *et al.* (1991) and Koicheva (1993) for the exponential distribution.

2. $c = \Gamma(p+k)a^{-k}/\Gamma(p)$ and $d(x) = \sum_{i=1}^{k} \frac{\Gamma(p+k)}{\Gamma(p+i)} a^{i-k-1} x^i$, we obtain the characterizations given in Adatia *et al.* (1991) and Koicheva (1993) for the Gamma distribution.

3. $c = 1/\gamma$ and $d(x) = x/\gamma k$, we obtain the characterization given in Ghitany *et al.* (1995) for the Weibull distribution.

4. $c = 1/\gamma$, $d(x) = x/2\gamma$ and $k = 2$, we obtain the characterization given in Ghitany *et al.* (1995) for the Rayleigh distribution.

5. $k = 1$, we obtain Theorem 3 given in Ruiz and Navarro (1994) for relations of type (1.5), as well as particular characterizations given in Kotz and Shanbhag (1980), Osaki and Li (1988), Ahmed (1991) and Nair and Sankaran (1991) for some usual distributions.

6. $d(x) = d \neq 0$, that is $m_k(x) = c + dr(x)$, where c and d are constant, then

$$f(x) = \alpha \exp\left\{\frac{c}{d}x - \frac{1}{d(k+1)}x^{k+1}\right\} \qquad \ldots (4.5)$$

where α is a constant. In particular, when $k = 1$, it characterizes the Normal distribution from relation $m_1(x) = \mu + \sigma^2 r(x)$, results given in Kotz and Shanbhag (1980) Nair and Sankaran (1991)

7. $d(x) = \sum_{i=0}^{k+1} a_i x^i$,we characterize the distributions belong to the Pearson family. Moreover, if $k = 1$, we obtain the characterization of Pearson family of distributions given in Nair and Sankaran (1991).

8. $k = 2$, $c = \alpha_2$ and $d(x) = 2x^2 + 2(2+n)x$ or $d(x) = (x^3 + nx)/(n-2)$, we characterize the Chi-Square and t-Student distributions, respectively.

For discrete distributions, we have the following theorem which has proof similar to that of Theorem 2.3.

THEOREM4.2. *Let X be a discrete r.v. with mass in $\{x_i : i = a, a+1, ..., b\}$, where $x_i < x_{i+1}$, a can be $-\infty$ and b can be ∞. Then, the following conditions are equivalent:*

1. $m_k(x_i) = c + d(x_i)r(x_i)$
2. $\triangle f(x_i)/f(x_i) = (c - \triangle d(x_i) - x_i^k)/d(x_{i+1})$

where, $f(x) = P(X = x), \triangle h(x_i) = h(x_{i+1}) - h(x_i)$, c is a constant, and $d(x)$ is a real function verifying

$$\lim_{i \to b} f(x_i)d(x_i) = 0 \qquad \ldots (4.6)$$

when $b = \infty$, or $d(x_b) = x_b^k - c$, when $b < \infty$.

REMARK 4.3. If in the above theorem, $\lim_{i \to a} f(x_i)d(x_i) = 0$ holds, then $c = \alpha_k$, where $\alpha_k = E(X^k)$. If we make $c = 0$, the above theorem proves that the ratio $m_k(x)/r(x)$, uniquely determines $F(x)$, in the discrete case.

REMARK 4.4. ¿From the above theorem, taking

1. $k = 1$, we obtain Theorem 4 given in Ruiz and Navarro (1994) for relations of type (1.5) as well as the characterizations given in Osaki and Li (1988) and Nair and Sankaran (1991).

2. $k = 2$. If $c = \alpha_2$ and $d(x) = qx(x + (n-1)(1-q))$ or $d(x) = x^2 + \lambda x$, we characterize the binomial and Poisson distributions, respectively. If $c = 0$ and $d(x) = ((1-a)x^2 + ax)/(1-a)^2$ we characterize the logarithmic series distribution.

3. $d(x_i) = d \neq 0$, that is $m_k(x_i) = c + dr(x_i)$, then

$$\frac{f(x_{i+1})}{f(x_i)} = \frac{c + d - x_i^k}{d} \qquad \ldots (4.7)$$

for all i. Hence, the discrete 0-1 Normal distribution must verify

$$\frac{f(x_{i+1})}{f(x_i)} = 1 - x_i \qquad \ldots (4.8)$$

Note that $x_i < 1$. However, it can be verified that a symmetric discrete distribution ($f(x) = f(-x)$, for all x) verifying (4.8) does not exist. In spite of this, there exist discrete distributions, such as that defined by

$$f(x_i) = \frac{1}{s} q^{(i^2)}$$

$$x_i = 1 - q^{2i+1}$$

where i is an integer, $q \in (0,1)$ and $s = \sum_{i=-\infty}^{\infty} q^{(i^2)}$, verifying (4.8). Moreover, it can be verified that $\mu = 0$, and, if $q = 2^{-1/2}$, then $\sigma^2 = 1$.

REMARK 4.5. Using the relation between $m_k(x)$ and $h_k(x)$, we can write the relations (1.6) using $h_k(x)$ and $r(x)$, both in continuous and discrete cases.

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