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A NOTE ON THE DISCRETE NORMAL DISTRIBUTION

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Abstract

Kemp [J. Statist. Plann. Inference 63 (1997), 223-229] studied a discrete analogue of the normal (Gauss) distribution extending the work of Dasgupta [Theo. Probab. Appl. 38 (1993), 520-524]. Kemp gives three characterizations of the discrete normal distribution including the characterization as the maximum entropy distribution (MED) in the integers for a given mean and a given variance. We reparametrize this model and obtain new properties, including two new characterizations. Moreover, we propose a generalization of this model and give several tables to calculate the associated distribution function and to approach other discrete distributions.

1. The Discrete Normal Distribution

Kemp [4] obtained the discrete normal distribution as the maximum entropy distribution (MED) with support in the integers, with specified

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mean and variance, and by using the Shannon entropy

$$H(X) = -\sum p(x)\log p(x),$$

where $p(x) = \Pr(X = x)$ is the probability mass function (pmf) for $x = \dots, -2, -1, 0, 1, 2, \dots$. A similar result can be obtained for the continuous normal distribution (see, for example, Kapur [3]). The model obtained by Kemp is

$$p(x) = \frac{\lambda^x q^{x(x-1)/2}}{\sum \lambda^x q^{x(x-1)/2}}, \quad x = \dots, -1, 0, 1, \dots, \tag{1}$$

where $\lambda > 0$ and $0 < q < 1$. This model includes Dasgupta's [2] model obtained by taking $\lambda = q^{1/2}$, characterized from a Cauchy equation on a discrete domain. The same pmf was obtained in a different way by Navarro et al. [6] on a different support and by using reliability properties. Bowman et al. [1] also obtained several discrete normal distributions from a discrete analogous to the Pearson family of distributions.

Model (1) can be reparametrized in a more usual form as

$$p(x) = \frac{\exp\{-(x-b)^2/2a^2\}}{c(a, b)}, \quad x = \dots, -1, 0, 1, \dots, \tag{2}$$

where

$$b = \frac{1}{2} - \frac{\log\lambda}{\log q}$$

$$a = +\sqrt{-1/\log q}$$

and

$$c(a, b) = \sum_{x=-\infty}^{\infty} \exp\{-(x-b)^2/2a^2\} < \infty.$$

Thus, X is symmetric with regard to b ($p(b-x) = p(b+x)$, for all x) if and only if, b is an integer or $b - 0.5$ is an integer. In the last case, X has a joint median in the interval $[b - 0.5, b + 0.5]$.

When b is an integer, we have the following properties.

Proposition 1. *If X has $p(x)$ equal to (2), where b is an integer, then*

1. $c(a, b) = c(a)$,

2. $E(X) = b$,

3. *the unique mode is* b,

4. $X - b$ *is a discrete normal distribution (verifies* (2)),

5. $(X - b)^2$ *has a Geometric distribution with support* $\{0, 1, 4, 9, \ldots\}$,

6. $Var(X) = a^3 c'(a)/c(a)$,

7. $\gamma_1(X) = 0$,

8. $\gamma_2(X) = \dfrac{3c(a)}{ac'(a)} + \dfrac{c(a)}{(c'(a))^2} c''(a) - 3$,

where $\mu = E(X)$, $\sigma^2 = Var(X)$, $\gamma_1(X) = E[(X - \mu)^3]/\sigma^3$ *and* $\gamma_2(X) = - 3 + E[(X - \mu)^4]/\sigma^4$.

Proof. The proofs of 1-5 and 7 are easy. From 1, we have

$$c(a) = \sum_{x=-\infty}^{\infty} \exp\{- (x - b)^2/2a^2\}$$

and by differentiating, we obtain

$$c'(a) = \sum_{x=-\infty}^{\infty} a^{-3}(x - b)^2 \exp\{- (x - b)^2/2a^2\}$$

and 6 holds. Differentiating again, we have

$$c''(a) = \sum_{x=-\infty}^{\infty} a^{-6}(x - b)^4 \exp\{- (x - b)^2/2a^2\}$$

$$- 3 \sum_{x=-\infty}^{\infty} a^{-4}(x - b)^2 \exp\{- (x - b)^2/2a^2\}$$

and hence, 8 holds.

Bowman et al. [1, p. 10] show that $\sigma^2(1) = 1$ and $\gamma_2(1) = 0$, but the proofs are erroneous because the model (2) does not belong to the discrete Pearson family defined in this paper by

$$\frac{p(x) - p(x-1)}{p(x-1)} = \frac{d - x}{a + bx + cx^2}.$$

However, we obtained the following asymptotic results.

Proposition 2. *If X has* $p(x)$ *equal to* (2), *where b is an integer, then*

1. $\lim_{a\to\infty} c(a)/a = \sqrt{2\pi}$,

2. $\lim_{a\to\infty} Var(a)/a^2 = 1$,

3. $\lim_{a\to\infty} \gamma_2(a) = 0$.

Proof. To obtain 1, we use that

$$\frac{1}{a}c(a) = \frac{1}{a}\sum_{x=-\infty}^{\infty} e^{-(x/a)^2/2}$$

(x integer) is an approximation for $\int_{-\infty}^{\infty} e^{-x^2/2}dx = \sqrt{2\pi}$, where the interval's length $1/a \to 0$. Hence, the condition $c(a)/a \to \sqrt{2\pi}$, implies $c'(a) \to \sqrt{2\pi}$, and from Proposition 1, $Var(a)/a^2 = ac'(a)/c(a) \to 1$ holds. Thus, from L'Hospital's rule $ac''(a)/c'(a) \to 0$ holds and hence, from Proposition 1, we obtain $\gamma_2(a) \to 0$.

These convergencies are very fast and we can approach $c(a)$ by $\sqrt{2\pi}a$ when $a > 0.7$. However, it is easy to obtain $\lim_{a\to 0} c(a) = 1$, and hence, the equality $c(a) = \sqrt{2\pi}a$ does not hold (see Table 1 in Section 3).

In the general case, that is when b is a real number, we obtain the following properties.

Proposition 3. *If X has* $p(x)$ *equal to* (2), *then*

1. $c(a, b) = c(a, b - [b])$, *where* $[b]$ *is the integer part of b.*

2. $c(a, b) = c(a, 1-b)$ *when* $0 \leq b \leq 1$.

3. $E(X-b) = \frac{1}{c(a, b)} a^2 \frac{\partial}{\partial b} c(a, b)$.

4. *If* $b - 0.5$ *is an integer, then* $E(X) = b$.

5. *If* $b - [b] < 0.5$, *then the unique mode is* $[b]$.

6. *If* $b - [b] > 0.5$, *then the unique mode is* $[b]+1$.

7. $E((X-b)^2) = \frac{1}{c(a, b)} a^3 \frac{\partial}{\partial a} c(a, b)$.

8. $E((X-b)^3) = 0$ *iff* $[b] = 0, 0.5$.

9. $E((X-b)^4)/E^2[(X-b)^2] = \frac{3}{a} \frac{c(a, b)}{\frac{\partial}{\partial a} c(a, b)} + \frac{c(a, b)}{\left(\frac{\partial}{\partial a} c(a, b)\right)^2} \frac{\partial^2}{\partial^2 a} c(a, b)$.

Proof. To obtain 1, we use that

$$c(a, b-[b]) = \sum_{x=-\infty}^{\infty} \exp\{-(x+[b]-b)^2/2a^2\}$$

$$= \sum_{y=-\infty}^{\infty} \exp\{-(y-b)^2/2a^2\}$$

$$= c(a, b).$$

The second property is obtained in a similar way.

To obtain 3, we note that

$$\frac{\partial}{\partial b} c(a, b) = \sum_{x=-\infty}^{\infty} \frac{(x-b)}{a^2} \exp\{-(x-b)^2/2a^2\}$$

$$= c(a, b) E(X-b)/a^2. \tag{3}$$

Property 4 holds when $b - 0.5$ is an integer since $p(b+x) = p(b-x)$ for all x.

Properties 5 and 6 are immediate.

To obtain Property 7 we note that

$$\frac{\partial}{\partial a} c(a, b) = \sum_{x=-\infty}^{\infty} \frac{(x-b)^2}{a^3} \exp\{-(x-b)^2/2a^2\}$$

$$= c(a, b)E((X-b)^2)/a^3. \tag{4}$$

The proof of Property 8 is similar to that of Property 4.

Finally, Property 9 is obtained from (3) and (4).

Remark 1. Note that from Property 3, if $c(a, b) = c(a)$ holds, then $E(X) = b$ and hence, $Var(X)$, $\gamma_1(X)$ and $\gamma_2(X)$ can be obtained from Properties 7, 8 and 9 in the preceding proposition. However, in general, this equality is not true (see Table 2). However, we have obtained the following asymptotic results.

Proposition 4. *If X has* $p(x)$ *equal to* (2), *then*

1. $\lim_{a\to\infty} c(a, b)/a = \sqrt{2\pi}$,

2. $\lim_{a\to\infty} \mu(a, b) = b$,

3. $\lim_{a\to\infty} V(a, b)/a^2 = 1$,

4. $\lim_{a\to\infty} \gamma_2(a, b) = 0$,

where $\mu(a, b) = E(X)$, $V(a, b) = Var(X)$ *and* $\gamma_2(a, b) = \gamma_2(X)$.

The proof of this proposition is similar to the proof of Proposition 2.

Remark 2. In particular, if $a > 0.7$, then $c(a, b) \simeq \sqrt{2\pi}a$, $Var(X) \simeq a^2$ and $\gamma_2(X) \simeq 0$ (see Tables 1 and 2). Szablowski [9] obtained exact expressions for $\mu(a, b)$ and $V(a, b)$ by using Jacobi Theta functions. From these expressions he obtained bounds for $|\mu(a, b) - b|$ and $|V(a, b) - a^2|$ when $a^2 > 1/\pi^2$.

2. Characterizations

In this section we study new characterizations for model (2) from reliability measures such as the failure rate (hazard function), the mean residual life or the truncated mean function. First we need a previous result from Navarro and Ruiz [7].

Proposition 5. *If X has a discrete distribution with support $\{..., x_{-1}, x_0, x_1, ...\}$ and $m(x, y) = E(X \mid x \leq X \leq y)$ is the doubly truncated mean function, then $m(x_k, x_{k+1})$ uniquely determines $p(x)$ by*

$$p(x_i) = \left[1 + \sum_{k<i} \lambda_k \cdots \lambda_{i-1} + \sum_{k>i} \frac{1}{\lambda_i \cdots \lambda_{k-1}}\right]^{-1} \text{ for all } i,$$

where

$$\lambda_k = \frac{x_{k+1} - m(x_k, x_{k+1})}{m(x_k, x_{k+1}) - x_k}.$$

As a consequence, we have the following characterization for the discrete normal distribution:

Proposition 6. *If X has a discrete distribution with support $\{..., -1, 0, 1, ...\}$, then $p(x)$ is equal to (2) if, and only if,*

$$m(k, k+1) = k + \frac{q^k}{d + q^k}, \text{ for all } k,$$

where $d > 0$ and $0 < q < 1$.

The most useful characterization for the continuous normal distribution using reliability measures was obtained in Kotz and Shanbhag [5]. They characterized the normal distribution by the relationship

$$m(x) = \mu + \sigma^2 r(x), \quad x \in \mathbb{R}, \tag{5}$$

where $r(x) = f(x)/(1 - F(x))$ is the failure rate function and $m(x) = E(X \mid X \geq x)$ is the left truncated mean function (which is equivalent to the mean residual life function $E(X - x \mid X \geq x) = m(x) - x$). Note that for the normal distribution neither $m(x)$ nor $r(x)$ has an explicit expression (both depend on $F(x)$). Relation (5) can be used to obtain a graphic plot test for normality. The general way to obtain F from a relation like $m(x) = c + g(x)r(x)$, was obtained by Ruiz and Navarro [8]. From this result, we obtain the following characterization for model (2).

Proposition 7. *If X has a discrete distribution with support $\{\ldots, -1, 0, 1, \ldots\}$, then $p(x)$ is equal to* (2) *if, and only if,*

$$m(k) = b + g(k)r(k), \quad \text{for all } k,$$

where

$$g(k+1) = (b - k + g(k))/(\lambda q^k),$$

$$g(0) = \sum_{x \geq 0} (x - b)\lambda^x q^{x(x-1)/2},$$

$q = e^{-1/2a^2}$ *and* $\lambda = q^{-b+1/2}$.

Note that the relations are different in the continuous case $(g = cte)$ and the discrete case. In Navarro et al. [6] we obtain a discrete distribution verifying

$$m(x_k) = \mu + \sigma^2 r(x_k)$$

with $p(x_k) = cq^{k^2}$, $x_k = \mu + \sigma^2(1 - q^{2k+1})$, $q = 1/\sqrt{2}$, $k = \ldots, -1, 0, 1, \ldots$. Thus, the definition of the discrete normal distribution can be extended by

$$p(x_k) = \frac{\exp\{-(k-b)^2/2a^2\}}{c(a, b)}, \tag{6}$$

where $\{x_k\}$ verifies $\cdots < x_{-1} < x_0 < x_1 < \cdots$ including both models. Hence, if X is a discrete normal using definition (6), then $aX + b$ is also a discrete normal distribution. Note that to obtain this last property we only need $x_k = \alpha + \beta k$.

3. Tables

In this section we present some tables and graphics for the model (2). From the tables, we obtain the following approximations:

(1) $c(a, b) \approx \sqrt{2\pi}a$ for $a \geq 0.7$

(2) $\mu(a, b) \approx b$ for $a \geq 0.7$

(3) $var(a, b) \approx a^2$ for $a \geq 0.7$

(4) $\gamma_2(a, b) \approx 0$ for $a \geq 0.7$.

However, we note that $\gamma_1(a, b)$ depends on b and it is zero only when $b = 0$ or $b = 0.5$ (see Table 2). Moreover, it is easy to show that $p(x, a, b) = p(x + n, a, b + n)$ for all integers n and that $p(x, a, b) = p(1 - x, a, 1 - b)$ for $0 < b < 1$.

Table 1. Values for $c(a, b)$ and $\sigma^2(a, b)$ when b is an integer (see Proposition 1)

a	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$c(a)$	1.0000	1.0000	1.0077	1.0879	1.2713	1.5064	1.7549	2.0053	2.2560
$\sigma^2(a)$	0.00	0.00	0.01	0.08	0.22	0.35	0.49	0.64	0.81
a	1	2	3	4	5	6	7	8	9
$c(a)$	2.5066	5.0133	7.5199	10.026	12.533	15.039	17.546	20.053	22.559
$\sigma^2(a)$	1	4	9	16	25	36	49	64	81
a	10	11	12	13	14	15	20	30	50
$c(a)$	25.066	27.573	30.079	32.586	35.093	37.599	50.133	75.199	125.33
$\sigma^2(a)$	100	121	144	169	196	225	400	900	2500

Table 2. Values for $c(a, b)$, $\mu(a, b)$, $\sigma^2(a)$ and $\gamma_1(a, b)$ when b is not an integer ($a = 1, 0.1$)

b	0	0.1	0.2	0.3	0.4
$c(1, b)$	2.5066	2.5066	2.5066	2.5066	2.5066
$\mu(1, b)$	0	0.1	0.2	0.3	0.4
$\sigma^2(1, b)$	1	1	1	1	1
$\gamma_1(1, b)$	0	0.0000008	0.0000013	0.0000012	0.000001
b	0.5	0.6	0.7	0.8	0.9
$c(1, b)$	2.5066	2.5066	2.5066	2.5066	2.5066
$\mu(1, b)$	0.5	0.6	0.7	0.8	0.9
$\sigma^2(1, b)$	1	1	1	1	1
$\gamma_1(1, b)$	0	–0.0000012	–0.0000017	–0.0000012	–0.0000009

b	0	0.1	0.2	0.3	0.4
$c(0.1, b)$	1	0.60653	0.13534	0.01111	0.00034
$\mu(0.1, b)$	0	0	0	0	0.00005
$\sigma^2(0.1, b)$	0	0	0	0	0.000045
$\gamma_1(0.1, b)$	0	485164256	3269023	22026	148
b	0.5	0.6	0.7	0.8	0.9
$c(0.1, b)$	0.00001	0.00034	0.01111	0.13534	0.60653
$\mu(0.1, b)$	0.5	0.99995	1	1	1
$\sigma^2(0.1, b)$	0.250000	0.000045	0	0	0
$\gamma_1(0.1, b)$	0	–148	–22026	–3269024	–485164224

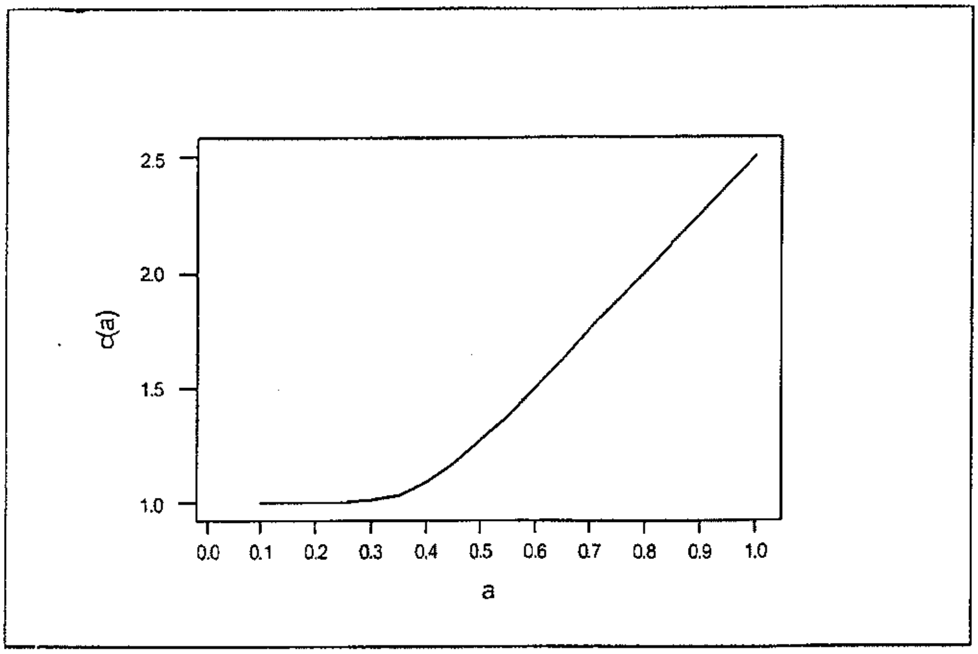

Figure 1. Values for $c(a, b)$ when b is an integer.

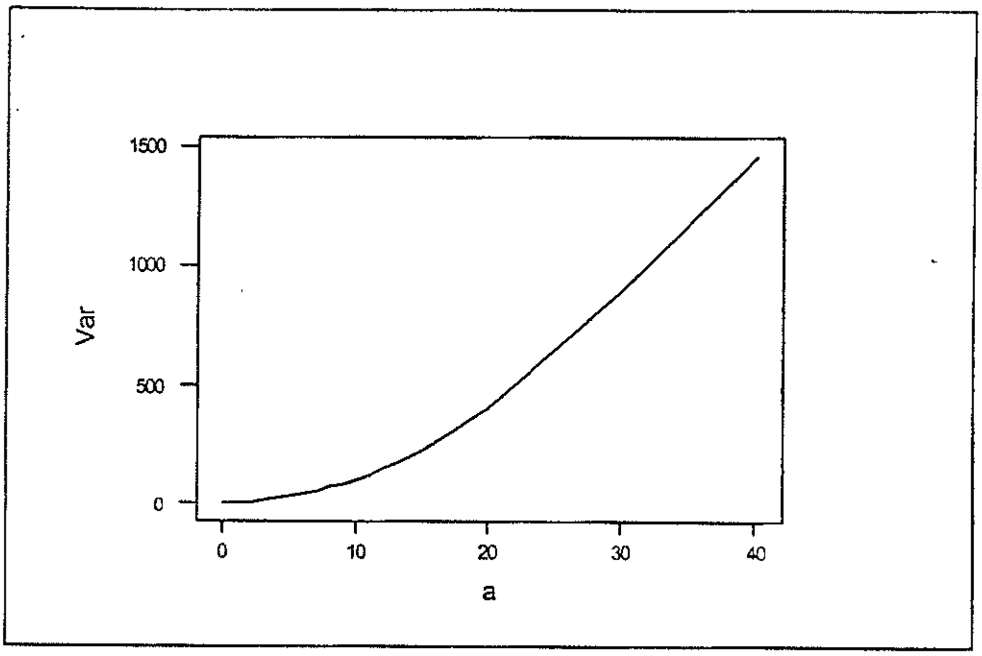

Figure 2. Values for $\sigma^2(a, b)$ when b is an integer.

Table 3. Distribution function $F(x, a)$ for the discrete normal $(b = 0)$

x	$a = 0.1$	$a = 0.2$	$a = 0.3$	$a = 0.4$	$a = 0.5$	$a = 0.6$	$a = 0.7$	$a = 0.8$	$a = 0.9$
0	1.0000	1.0000	0.99616	0.95961	0.89329	0.83191	0.78492	0.74934	0.72163
1			1.0000	1.0000	0.99974	0.99743	0.99032	0.97765	0.96074
2					1.0000	1.0000	0.99994	0.99956	0.99826
3							1.0000	1.0000	0.99998
4									1.0000

x	$a = 1$	$a = 2$	$a = 3$	$a = 4$	$a = 5$	$a = 6$	$a = 7$	$a = 8$	$a = 9$
0	0.69947	0.59974	0.56649	0.54987	0.53990	0.53327	0.52859	0.52519	0.52268
1	0.94144	0.77577	0.69228	0.64653	0.61811	0.59888	0.58520	0.57519	0.56775
2	0.99543	0.89675	0.79877	0.73455	0.69176	0.66182	0.64009	0.62403	0.61200
3	0.99986	0.96151	0.87942	0.80984	0.75841	0.72053	0.69226	0.67100	0.65491
4	1.00000	0.98851	0.93409	0.87033	0.81635	0.77381	0.74083	0.71547	0.69600
5		0.99727	0.96725	0.91599	0.86475	0.82082	0.78514	0.75691	0.73487
6		0.99949	0.98525	0.94837	0.90359	0.86117	0.82475	0.79495	0.77118
7		0.99992	0.99399	0.96994	0.93353	0.89486	0.85943	0.82931	0.80470
8		0.99999	0.99779	0.98344	0.95572	0.92221	0.88919	0.85987	0.83525
9		1.00000	0.99927	0.99137	0.97151	0.94381	0.91421	0.88664	0.86276
10			0.99978	0.99575	0.98231	0.96040	0.93483	0.90971	0.88723
11			0.99994	0.99803	0.98940	0.97280	0.95146	0.92928	0.90872
12			0.99999	0.99914	0.99388	0.98180	0.96462	0.94564	0.92736
13			1.00000	0.99964	0.99660	0.98816	0.97481	0.95910	0.94334
14				0.99986	0.99818	0.99254	0.98255	0.97000	0.95687
15				0.99995	0.99907	0.99546	0.98831	0.97869	0.96818
16				0.99998	0.99955	0.99736	0.99251	0.98550	0.97752
17				0.99999	0.99979	0.99856	0.99550	0.99077	0.98514
18				1.00000	0.99991	0.99930	0.99760	0.99478	0.99128
19					0.99997	0.99974	0.99903	0.99779	0.99616
20					1.00000	1.00000	1.00000	1.00000	1.00000

Remark 3. Note that, in Table 3, we can use that

1. $F(x, a) = F(-x-1, a)$

2. $F(x, a, b) = F(x-b, a)$ when b is an integer

3. $|F(x, a, b) - F([x-b], a)| < 0.04$ when $b - [b] \geq 0.5$ and $a > 5$

4. $|F(x, a, b) - F([x-b]+1, a)| < 0.04$ when $b - [b] \leq 0.5$ and $a > 5$.

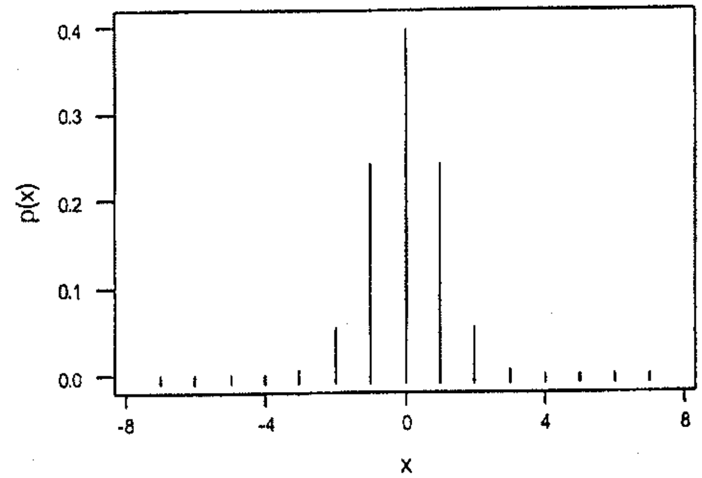

Figure 3. Values for $p(x)$ in a standard discrete normal.

4. Approaching Distributions

First, we show that the discrete normal distribution can be used to approach the continuous normal distribution and vice versa, with a continuity correction.

Proposition 8. *If X is a discrete normal* $N(a, b)$ *defined by* (2) *and* Y *is a continuous normal* $N(\mu = b, \sigma = a)$, *then*

$$\lim_{a\to\infty} |F_X(x, a, b) - F_Y(x+0.5, a, b)| = 0$$

for all integers x. *If* $a \geq 1$, *then* $F_X(x, a, b) \approx F_Y(x+0.5, a, b)$ *with a* *maximum difference* $|F_X(x, a, b) - F_Y(x + 0.5, a, b)| \leq 0.01072$. *We show in the next table the distribution functions for both standard normal distributions.*

x	0	1	2	3	4	MaxDif
$F_X(x, 1, 0)$	0.69947	0.94144	0.99543	0.99986	1.00000	0.00825
$F_Y(x + 0.5, 1, 0)$	0.69146	0.93319	0.99379	0.99977	1.00000	-

Thus, we can also use the discrete normal distribution to approach other distributions (Binomial, Poisson, Negative Binomial, etc.). For example, if we approach a Binomial $Y \equiv B(n = 30, p = 0.5)$ using a discrete normal $X \equiv N(a = 2.73, b = 15)$, we obtain

x	15	16	17	18	19	20
$p_X(x)$	0.145673	0.136278	0.111575	0.079947	0.050134	0.027514
$p_Y(x)$	0.144464	0.135435	0.111535	0.080553	0.050876	0.027982
x	21	22	23	24	25	MaxDif
$p_X(x)$	0.013215	0.005555	0.002044	0.000658	0.000185	0.001209
$p_Y(x)$	0.013325	0.005451	0.001896	0.000553	0.000133	-

In this way, we have obtained the following result.

Proposition 9. *If X has* $p_X(x, a, b)$ *equal to* (2) *and Y has a Poisson distribution with* $p_Y(x, \lambda) = e^{-\lambda}\lambda^x/x!$, *then*

$$\lim_{\lambda \to \infty} \frac{p_X(\lambda + m, a = \sqrt{\lambda}, b = \lambda)}{p_Y(x + m, \lambda)} = 1,$$

where m is a fixed integer (it does not depend on λ*).*

Proof. From the definitions, we have

$$\lim_{\lambda\to\infty} \frac{p_X(\lambda + m,\ a = \sqrt{\lambda},\ b = \lambda)}{p_Y(\lambda + m,\ \lambda)} = \lim_{\lambda\to\infty} \frac{e^{-\lambda}e^{-m^2/2\lambda}(\lambda + m)!}{\sqrt{2\pi\lambda}\lambda^{\lambda+m}}$$

which, using Stirling's formula, is equal to

$$\lim_{\lambda\to\infty} \frac{e^{-m^2/2\lambda}(\lambda + m)\cdots(\lambda + 1)}{\lambda^m} = 1.$$

In Tables 4 and 5, we give the maximum differences between some discrete probability mass functions $(p(x))$ and the corresponding discrete normal probability mass functions.

Table 4. Maximum differences between the discrete normal and Poisson $P(\lambda)$ and Binomial $B(n, p)$ pmfs

$P(\lambda)$ λ	MaxDif	$B(n, p)$ p	MaxDif $n = 30$	n	MaxDif $p = 0.5$	npq	MaxDif $p > 0.01$ $p < 0.99$
0.5	0.167139	0.01	0.110586	1	0.0160584	1	0.122821
1	0.125909	0.02	0.162418	2	0.0641898	2	0.050439
1.5	0.069264	0.03	0.132518	3	0.0222131	3	0.029743
2	0.050975	0.04	0.094905	5	0.0111758	4	0.024655
5	0.020780	0.05	0.062885	7	0.0073451	5	0.020209
7	0.014404	0.07	0.046569	10	0.0062196	6	0.016791
10	0.009638	0.1	0.025911	15	0.0028785	7	0.014186
15	0.006550	0.2	0.012480	20	0.0022154	8	0.012175
20	0.004764	0.3	0.006144	25	0.0014361	9	0.010593
30	0.003191	0.4	0.002694	30	0.0012087	10	0.009326
50	0.001887	0.5	0.001209	50	0.0005628	20	0.004710

Table 5. Maximum differences between the discrete normal and Negative Binomial $NB(n, p)$ pmfs

p	MaxDif $n = 5$	MaxDif $n = 10$	n	MaxDif $p = 0.5$	μ	MaxDif $n = 5$	MaxDif $n = 1$
0.1	0.005179	0.002338	2	0.129015	0.5	0.192348	0.276727
0.2	0.011053	0.004984	3	0.070800	1	0.161793	0.280304
0.3	0.017702	0.008077	4	0.046402	2	0.069221	0.216634
0.4	0.026440	0.011891	5	0.036746	3	0.058772	0.170848
0.5	0.036746	0.016574	6	0.030223	5	0.036746	0.118650
0.6	0.051829	0.023770	7	0.025226	7.5	0.026440	0.085505
0.7	0.067895	0.034593	8	0.021395	10	0.020265	0.066765
0.8	0.134103	0.057300	9	0.018412	20	0.011053	0.035528
0.9	0.195038	0.130905	10	0.016574	40	0.005793	0.018342

Note that to have a maximum difference smaller than 0.05, we can approach a Poisson with a mean bigger than 2, a Binomial, with $npq > 2$ and $0.01 < p < 0.99$, a Geometric with a mean bigger than 10 and a Negative Binomial with a mean bigger than 5 and $n > 5$. Similar differences can be obtained using the distribution functions.

References

[1] K. O. Bowman, L. R. Shenton and M. A. Kastenbaum, Discrete Pearson distributions, Oak Ridge National Laboratory Technical Report, TM-111899, Oak Ridge, TN, 1991.

[2] R. Dasgupta, Cauchy equation on discrete domain and some characterization theorems, Theo. Probab. Appl. 38 (1993), 520-524.

[3] J. N. Kapur, Maximum Entropy Models in Science and Engineering, Wiley Eastern, New Delhi, India, 1989.

[4] A. W. Kemp, Characterizations of a discrete normal distribution, J. Statist. Plann. Inference 63 (1997), 223-229.

[5] S. Kotz and D. N. Shanbhag, Some new approaches to probability distributions, Adv. Appl. Probab. 12 (1980), 903-921.

[6] J. Navarro, M. Franco and J. M. Ruiz, Characterization through moments of the residual life and conditional spacings, Sankhya 60 (1998), 36-48.

[7] J. Navarro and J. M. Ruiz, Characterization of discrete distributions using expected values, Statist. Papers 36 (1995), 237-252.

[8] J. M. Ruiz and J. Navarro, Characterization of distributions by relationships between failure rate and mean residual life, IEEE Trans. Reliability 43 (1994), 640-644.

[9] P. J. Szablowski, Discrete normal distribution and its relationship with Jacobi theta functions, Stat. Probab. Lett. 52 (2001), 289-299.

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